On robust cycle bases

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Abstract

Two types of robust cycle bases are defined via recursively nice arrangements; complete and bipartite complete graphs are shown to have such bases. It is shown that a diagram in a groupoid is commutative up to natural equivalence (cutne) if for each cycle in a robust basis of the graph underlying the diagram, the composition of the morphisms is naturally equivalent to the identity. For a hypercube Q_n, it is shown that the commutativity (or cutne) of a particular subset of asymptotically $4/n$ of the square faces forces commutativity (or cutne) of the entire diagram.
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1 Introduction

We study the elementary properties of a new type of cycle basis for graphs based on the possibility of ordering those basis cycles which sum to a given cycle in such a way that some recursive condition holds.

A cycle in some graph is a subset of the edges that induces a connected subgraph which is regular of degree 2. A binary operation can be defined on the subsets of any set using symmetric difference or mod-2 sum; the sum is the

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union of the subsets with elements appearing twice deleted. It is easy to check that this operation is commutative and associative. The Eulerian circuits (i.e., subsets of the edges which induce subgraphs having no odd degree vertices) are sums of cycles and conversely. We will be interested in conditions that ensure the sum of cycles is another cycle.

A *cycle basis* is a subset $\mathcal{B}$ of the set $Cyc(G)$ of all cycles in G having the minimal spanning property that for any element Z in $Cyc(G)$ there is a unique subset $\mathcal{C}(G, \mathcal{B}, Z) = \mathcal{C}(Z) = \mathcal{C} \subseteq \mathcal{B}$ such that Z is the sum $\sum \mathcal{C}$ of the elements in $\mathcal{C}$. Since the resulting concept is identical to the usual algebraic description of a cycle basis as a maximum linearly independent subset of a certain vector space, all cycle bases for a particular graph G have the same number of elements, $b(G)$. If G has p vertices, q edges and k connected components, then $b(G) = q - p + k$. In particular, if G is any connected graph and T is a spanning tree, then there is a basis of G corresponding to T which consists of the cycles $e \cup P(e)$, where e is an edge of G not in T and $P(e)$ is the unique path in T between the endpoints of e.

Cycle bases of various special types have long been studied. In 1937, Mac Lane showed that for a 2-connected graph, planarity is equivalent to the existence of a 2-*basis*, which is a cycle basis in which every edge belongs to at most two cycles from the basis [5]. For any 2-connected plane graph, the cycles determined by the boundaries of the finite regions constitute a 2-basis.

Every 2-connected graph can be constructed by beginning with a cycle and adding on new cycles in such a way that each new cycle intersects the union of those that have preceded it in a nontrivial path. See, e.g., Lovasz [4]. The condition seems natural and the following definition uses an analogous nontrivial-path-intersection property to constrain the way a cycle basis represents the cycles.

A cycle basis $\mathcal{B}$ is *robust* if for every cycle Z there is a linear ordering of the subset $\mathcal{C}(G, \mathcal{B}, Z)$ such that, as each element of the resulting sequence is added to form the sum Z, it intersects the *sum* of those preceding in a nontrivial path. In this case, the partial sums must be cycles. A cycle basis is called *cyclically robust* when the sum of the new cycle and those that went before remains a cycle.

The term "robust" is used to indicate that the property of being a cycle is preserved by the partial sums or even that the next basis cycle is meeting the previous partial sum in a nice way.

A diagram is a directed graph that lives in a category; i.e., the vertices are objects of the category and the arcs are morphisms. See, e.g., Mac Lane [7]; a brief review is in section 3 below.

We apply robustness to questions of diagram coherence related to Mac Lane

[6] and Stasheff [11]. They asked when the commutativity of all diagrams of a certain type is implied by that of a few particular cases just as associativity for n-term expressions follows from that of the 3-term expressions. Instead, we study when the commutativity of a diagram can be inferred from the knowledge that a particular subset of the faces is commutative.

To utilize cycles, our proofs require that the morphisms be invertible; that is, we assume that the categories are groupoids. A cycle in the underlying graph of a diagram in a groupoid commutes if the composition of all its morphisms (or their inverses) is the identity (no matter where you start). It commutes up to a natural equivalence if one of the compositions going around the cycle is naturally equivalent to the identity; equivalently, for any way to divide the cycle into two parallel paths, there is a natural equivalence between them. Diagrams are said to commute (or commute up to a natural equivalence) if every one of their cycles does. Similar results hold if one considers diagrams which commute up to a homotopy.

Using the fundamental group, we show that a diagram commutes if every cycle in a basis commutes (Theorem 1 of section 3). However, this may not be true if instead of (strict) commutativity, one considers commutativity up to natural equivalence. In Theorem 2 of section 3, we show that diagrams commute up to natural equivalence (or homotopy) if the cycles of a *robust* basis do.

Diagrams based on the hypercube Q_n are dealt with separately. A cycle basis is introduced which is conjectured to be robust. It is shown directly that various coherence results hold for the hypercube. Commutativity (or generalized commutativity) can be *forced* by that of a particular subset of the faces. We further, investigate when global commutativity is guaranteed if the number of possibly noncommutative faces is reduced below a certain "blocking" threshold, which we conjecture to be $n - 1$.

The paper is organized as follows. In the next section, we define robust cycle bases and show that they exist for all graphs. Section 3 reviews the notions of category, groupoid, diagram and natural equivalence in order to show how robustness can be applied to the theory of commutativity. In Section 4, we consider the special case of the hypercube. Section 5 contains some remarks.

2 Cycle bases

A cycle basis for a graph G is a set $\mathcal{B}$ of cycles in the graph with the property that any cycle Z in the graph is the sum of a unique nonempty subset $\mathcal{C}$ of $\mathcal{B}$.

A particular ordering of this subset is called a *sum sequence* for Z. A sequence of cycles $Z_1, \ldots, Z_k$ is *well-arranged* if for all j $(2 \leq j \leq k)$, the intersection of the $j-th$ cycle with the preceding partial sum of all cycles up to the $j-1$-st is a nontrivial path: $Z_j \cap \sum_{i=1}^{j-1} Z_i \approx P_2$, where "$\approx$" denotes homeomorphism and P_2 is the path of two vertices. A cycle basis is *robust* if every cycle has a well-arranged sum sequence.

A sequence $Z_1, \ldots, Z_k$ of cycles is called *cyclically well-arranged* if the partial sums $Z_1 + \cdots + Z_j$ are all cycles. A cycle basis is *cyclically robust* if every cycle has a cyclically well-arranged sum sequence. Well-arranged sequences are automatically cyclically well-arranged but the reverse may not hold.

Indeed, let K_4 be linearly embedded in the plane, let T_1, T_2, T_3 be the three triangles (K_3 subgraphs) determined by the bounded faces and let Z_1, Z_2, Z_3 be the quadrilaterals formed by summing two of the three triangles. Then the sum of any two of the Z_j quadrilaterals is the third but any two of the Z_j intersect in a pair of disjoint edges.

Not every cycle basis is robust. The following example is due to A. Vogt. Let G be the graph consisting of a 6-cycle, with an inscribed triangle; this graph has 6 vertices and 9 edges so a cycle basis must have 4 elements. Let $\mathcal{B}$ consist of the three diamond-shaped 4-cycles, each determined by two of the edges in the inscribed triangle and the opposite two edges of the hexagon. The inscribed triangle is the 4-th cycle. It is easy to check that the hexagon, while it is the sum of the three diamonds, has no sum sequence that is cyclically well-arranged. However, G does have a robust basis obtained by taking the four bounded triangular faces.

Examples of robust bases can be given. For planar graphs, if we take the boundaries of the bounded regions, then the resulting cycle basis is easily seen to be robust. Bases are given below for the complete and bipartite complete cases. For the hypercube, a possibly robust basis will be constructed.

For the complete graph K_n, with vertices $1, \ldots, n$, let $\mathcal{B}_n$ be the set of all triangles with n as one of their vertices.

Proposition 1 *For every positive integer n, $\mathcal{B}_n$ is a robust basis for K_n.*

Proof. Suppose $Z = i_1, \ldots, i_r$ is any cycle in K_n. If n does not belong to Z, then $Z = (n, i_1, i_2) + \cdots (n, i_{r-1}, i_r) + (n, i_r, i_1)$ is a well-arranged sum sequence; each basis triangle after the first intersects the partial sum in a single edge, except for the last triangle which intersects in two consecutive edges. If n is in the cycle, we may assume WLOG that it is last: $i_r = n$. Then $Z = (n, i_1, i_2) + \cdots + (n, i_{r-2}, i_{r-1})$ is the sum sequence. □

By a similar method, one can prove the following:

Proposition 2 *For every pair of positive integers p,q, $K_{p,q}$ has a robust basis.*

The robust basis consists of all 4-cycles containing two fixed vertices, one of each color.

When do robust bases exist? A. Vogt proposed heuristically to take short cycles for the basis. This could be carried out via a greedy algorithm - or one might look for minimum total length. Such a strategy is certainly satisfied by $\mathcal{B}_n$ which consists only of 3-cycles and the robust basis for the bipartite complete graph has only 4-cycles.

If the following conjecture holds, then the hypercubes also have a robust basis consisting only of 4-cycles.

Conjecture 1 *If G is any graph with a robust basis $\mathcal{B}$ and T is any tree, then $G \times T$ has a robust basis obtained by adding to $\mathcal{B}$ the 4-cycles determined by the cartesian products of the edges of G with the edges of T.*

This is close to being a theorem but the proof still involves some hand-waving. Our original version was for $T = K_2$; R. Jamison suggested the generalization to T.

M. M. Shikare wondered if every basis which comes from a spanning tree of a connected graph must be robust. However, M. Ostrowskii has given an example to show that this is not true: Take K_5 as the graph and let the tree be P_5. Let Z be the five-cycle obtained by extending the path to a cycle and then taking the complement.

3 Bases and commutativity

Let $D = (V, A)$ be a directed pseudomultigraph; write $s(a)$,$t(a)$, resp., to indicate the source and target of the arc a, so $a = (s(a), t(a))$. The arc-pair a, a' is *composable* if $s(a) = t(a')$. A function Φ from the set of all composable arc-pairs to A will be called a *law of composition* if $s(\Phi(a, a')) = s(a')$ and $t(\Phi(a, a')) = t(a)$. The pair (D, Φ) constitute a *category* if the law of compostion is associative and has right and left identities. In a category, vertices are called *objects* (e.g., topological spaces or groups, etc.) and arcs *morphisms* (e.g., continuous maps or homomorphisms). Morphisms were originally only set-functions that preserved some structure so composition is usually written right-to-left as for compostion of functions. If a morphism has a two-sided inverse, it is called an *isomorphism* and said to be *invertible.*

A *diagram* in a category is a directed graph whose vertices are objects and whose arcs are morphisms. Equivalently, a diagram is a subdigraph of the underlying digraph of the category.

Intuitively, diagrams commute when directed v-w-paths induce a well-defined morphism from v to w, and this can be generalized to allow some sort of relation, not necessarily equality, between pairs of such morphisms. For example, different types of generalized commutativity apply to diagrams of topological spaces and continuous maps based on homotopy or isotopy. It would be interesting to formalize the notion of generalized commutativity but here we only consider the special cases of commutativity and of commutativity up to natural equivalence for diagrams in which all morphisms are equivalences (i.e., have two-sided inverses).

A category is called *small* if the collection of its objects is a set. (This is a technical restriction designed to avoid problems like the Russell paradox, so the reader can essentially ignore it.) A *groupoid* is a small category in which all morphisms are invertible. If $\mathcal{G}$ is a groupoid, then it includes the inverse of each morphism. For simplicity, we assume further that all diagrams are finite.

A cycle in the graph underlying a diagram in a groupoid is defined to be *commutative* if the composition of the morphisms in either of the two directed cycle orientations is the identity, and this property is independent of base point as well. A diagram is called commutative if and only if all of its cycles are.

The usual notion of commutativity for diagrams is that if two directed paths have the same initial and terminal object, then the resulting composition of morphisms should be independent of the path chosen. There is a slight technical distinction since the usual notion of commutativity does not place any restriction on the diagram consisting of a single cycle with an alternating sequence of arcs. Our groupoid condition requires that the composition of all the morphisms (and their inverses when the arcs are traversed backward) must be the identity if you go around a cycle.

Theorem 1 *Let D be any finite diagram in a groupoid $\mathcal{G}$. Then D is commutative if it has a basis of commutative cycles.*

Proof. Let $\mathcal{B}$ be a basis of commutative cycles for the diagram D; that is, for the underlying graph G. The fundamental group of the topological realization of G is the free group on $\mathcal{B}$. See, e.g., Spanier [10]. Hence, any cycle Z in G can be expressed as a word in the generators and their inverses. Since the letters all evaluate to the identity because of our assumption about $\mathcal{B}$, the words must also evaluate to the identity so the cycle Z commutes. $\square$

A basis of cycles for the hypercube Q_n, for instance, contains $1 + (n-2)2^{n-1}$ elements, while there are a total of $n(n-1)2^{n-3}$ square faces and exponentially many more longer ones depending on permutations. Hence, according to the theorem, only a very sparse set of cycles suffices to determine the commutativity of a much larger set.

Given a cycle Z in a diagram and any two vertices v, w in Z, there is a unique orientation of the edges of the cycle to provide two directed paths from v to w. Note that there is no way to to say which of the two paths is first unless the cycles are assumed to have a given clockwise or counterclockwise sense. Let $Z(v,w)$ denote the unordered pair of directed paths. Each path induces a morphism in the groupoid. These morphisms agree if and only if the cycle commutes.

We now consider a special case for concreteness, but it is clear that the arguments can be substantially generalized. A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function F assigning to each object c of $\mathcal{C}$ an object $F(c)$ in $\mathcal{D}$ and to each morphism $\alpha : c \to c'$ in $\mathcal{C}$ a morphism $F(\alpha) : F(c) \to F(c')$ such that F preserves the law of composition. Denote by **CAT** the category whose objects are small categories with functors as the morphisms and functional composition.

Just as functors are morphisms between categories, a *natural transformation* is a morphism between functors. For functors F_1, F_2 both from $\mathcal{C}$ to $\mathcal{D}$, a natural transformation ν is a function associating to each object c in C a morphism $\nu_c : F_1 c \to F_2 c$ in D which is compatible with the functors F_1, F_2 in the sense that for all $\alpha : c \to c'$ in C, the following holds:

$$\nu_{c'} \circ F_1 \alpha = F_2 \alpha \circ \nu_c.$$

Natural transformations may be composed and a natural transformation with a two-sided inverse is called a natural *equivalence*.

Let $\mathcal{G}$ be any groupoid subcategory of **CAT**; that is, the objects of $\mathcal{G}$ constitute a set of categories and the morphisms are invertible functors. When natural transformations consist only of invertible mappings, it is easily checked that the resulting inverse transformation must be natural, so they are actually natural equivalences.

A still more general notion than category is 2-category (see, e.g., [1]), where multiple laws of composition also obey rules of mutual consistency. However, we shall now return to the connection of these ideas with cycles.

Let D be any diagram in a groupoid subcategory of **CAT**. A cycle is *commutative up to natural equivalence* if given any two distinct vertices there is a natural equivalence between the two functors induced by the paths in $Z(v,w)$. Since the identity is a natural equivalence, commutativity is a special case. It is easy to check that the existence of a natural equivalence does not depend on the choice of vertices; this allows us to make a composable choice of natural equivalences when two cycles meet in a nontrivial path.

Lemma 1 *Suppose that two cycles in the underlying graph of a diagram in-*

tersect in a nontrivial path. If both cycles are commutative up to a natural equivalence, then so is their sum.

Proof. Suppose the cycles Z_1, Z_2 intersect in the nontrivial path P. Since Z_1 commutes, there is a path $P_1' = Z_1 - P$ and a natural equivalence ν_1 from P_1' to P. Similarly, there is a path $P_2' = Z_2 - P$ and a natural equivalence ν_2 from P_2' to P. So the composition $\nu_2^{-1} \circ \nu_1$ is an equivalence of P_1' with P_2', which means that $Z_1 + Z_2$ commutes. $\square$

A diagram is commutative up to natural equivalence if and only if all of its cycles are. By applying the lemma recursively to a robust cycle basis, any cycle of the diagram can be checked for commutativity up to natural equivalence.

Theorem 2 *Let D be a diagram in a groupoid subcategory of* **Cat** *and suppose that there is a robust cycle basis for D consisting only of cycles which commute up to natural equivalence. Then D commutes up to natural equivalence.*

As noted below, substantially less than a robust basis may be needed if the recursive procedure is made more intricate.

4 Forcing sets and blocking numbers for hypercubes

We define a basis $\mathcal{R}_n$ for the hypercube Q_n of dimension $n \geq 2$ as follows: $\mathcal{R}_2$ is the cycle Q_2 and having recursively given $\mathcal{R}_{n-1}$ as a subset of the 4-cycles of Q_{n-1} which is embedded in Q_n as the set of all nodes with first coordinate equal to 0, $\mathcal{R}_n$ is obtained by including the $(n-1)2^{n-2}$ 4-cycles of the form $0s, 0t, 1t, 1s$, where st is any edge in Q_{n-1}. It is straightforward to check that $\mathcal{R}_n$ is independent and has $b(Q_n)$ elements so is a basis. Indeed, we believe it is a robust basis but as yet have no proof.

For n equal to 3, this basis consists of 5 of the 6 faces. The following is a special case of the category theory result known as the **cube lemma**; see, e.g., [8].

Lemma 2 *Let D by any diagram in a groupoid subcategory of* **CAT** *with underlying graph Q_3. If five of the six faces commute up to a natural equivalence, then so does the sixth.*

This lemma is a consequence of Theorem 2, and gives the following result, which also follows from Theorem 2 if $\mathcal{R}_n$ is robust.

Theorem 3 *For $n \geq 2$ the hypercube Q_n commutes up to natural equivalence if every cycle in $\mathcal{R}_n$ is commutative up to natural equivalence.*

Proof. First, note that if all the 4-cycles (i.e., the 2-dimensional faces) are commutative up to natural equivalence "cutne" then so is Q_n. We sketch an elementary argument similar to Gray [1] who showed that, as a 2-category, Q_n commutes if and only if all of its Q_3 faces do.

It is enough to consider cycles formed from two essentially disjoint v-w-paths which are geodesic of length j ($2 \leq j \leq n$). These paths correspond to permutations on n coordinates and by the transitivity of the symmetric group, there is a sequence of transpositions carrying the first permutation to the second. Each transposition corresponds to a set of squares so the commutativity up to natural equivalence of the squares provides a sequence of natural equivalences whose composition is a natural equivalence between the two paths.

Hence, it suffices to show that every square face f on the front Q_j face of Q_{j+1} is cutne, and this follows by applying the cube lemma to the unique 3-dimensional cube c having f as its front face. By induction, the back face of c is cutne and the side faces are assumed to be cutne. □

Recalling our previous remark, we obtain a *forcing set* for Q_n with $1+(n-2)2^{n-1}$ elements.

Let $\mathcal{C}$ be a category. Define the *blocking number* $\beta(Q_n)$ of the hypercube Q_n with respect to the category $\mathcal{C}$ to be the smallest possible size of any subset of the Q_2-faces which can fail to be cutne for all possible non-cutne diagrams in $\mathcal{C}$ on the scheme Q_n.

In trivial cases, e.g., one object and all morphisms the identity, then no diagram could fail to commute. Call a category *nontrivial* if it contains at least one object with a nonidentity morphism.

Note that of course a single noncommutative face means that Q_n can't be commutative but, in fact, when commutativity fails, it must do so multiple times. The cube lemma implies that $\beta(Q_3)$ is at least 2 and, in fact, equality holds. More generally, we believe the following holds.

Conjecture 2 *Let $\mathcal{G}$ be any nontrivial groupoid subcategory of* **CAT**. *For n at least 2, $\beta(Q_n) = n-1$ with respect to $\mathcal{G}$.*

5 Remarks

Even if a basis is not robust, it may have the property of containing well-arranged sum sequences for almost every cycle. Thus, from the commutativity up to natural equivalence (cutne) of such a basis, nearly every cycle would be cutne.

Vogt observes that to show a diagram is cutne, it is enough to show every cycle has a well-arranged sum sequence taken from a set of cycles that have already been shown to be cutne. For instance, if the cycles in the nonrobust basis given above in section 2 are cutne, then so are the three triangular cycles different from the inscribed triangle. Hence, the hexagon is cutne as well. Indeed, given any set of cycles, we can form the closure with respect to well-arranged sequences. How large is the closure?

The requirement of invertible morphisms includes many interesting special cases. For example, in the case of quantum computing, all operations are reversible and hence invertible. Also, neural networks have been used to model invertible maps [9].

Robustness is a general theme which appears in many areas of mathematics. Hyers and Ulam [2] discovered that if two spaces are within a sufficiently small tolerance of being isometric, then they are in fact isometric (and the isometry is globally close to the near-isometry). A robustness phenomenon also occurs with respect to orthogonality. We showed with Kurkova [3] that in n-dimensional Euclidean space, if any two of a set of unit vectors have dot product not exceeding $1/n$ in absolute value, then the set cannot have more than n members.

We wonder if these results could be applied in cognitive science since they show how commutativity of a diagram (global information) is not affected by possible failure of a square to commute (local error). Many mathematical properties, for example, can be described via commutativity of diagrams.

The results here imply that for diagrams in Q_n the fraction of the square faces which need to be checked to infer commutativity is asymptotically approaching $4/n$ provided that one chooses faces corresponding to the cycles in the recursive basis. Hence, increasing complexity of the diagram can provide a larger payoff for the "intelligent" checking of commutativity.

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