Power and legitimacy in pillage games

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March 12, 2006

Abstract

A *pillage game* is a formal model of Hobbesian anarchy as a coalitional game. The technology of pillage is specified by a power function that determines the power of each coalition as a function of its members and their wealth. A coalition can despoil any other coalition less powerful than itself. The present paper studies the extent to which the exercise of power can be constrained by a shared concept of legitimacy. The basic pillage game is augmented by a set of extrinsic variables that can convey information about past behavior. Depending on the power function, the illegitimate use of power can be inhibited by legitimizing the subsequent use of power against the transgressors. Legitimacy is modeled in a static sense, called *quasi-legitimacy*, using the stable set (von Neumann-Morgenstern solution) of the augmented pillage game, and in an explicitly dynamic sense, called simply legitimacy, using a concept of farsighted core. Quasi-legitimacy is shown to be a necessary but not sufficient condition for legitimacy. The sets of quasi-legitimate wealth allocations are characterized, and an iterative process is developed for constructing the largest quasi-legitimate set of allocations for each pillage game. If the power function gives enough weight to coalition size that no individual can be as powerful as the coalition of everyone else, then a natural augmentation of the basic pillage game can legitimize the set of all allocations. However, if the power of each coalition is determined by its total wealth alone, then even the weaker concept of quasi-legitimacy cannot stabilize anything other than the stable set of the basic pillage game.

1. Introduction

A pillage game is a natural setting for the study of power. There is only one commodity, wealth, which is allocated among a finite number of players. Since wealth is desired by all, reallocation can only be effected by force. The technology of force is specified by a power function that determines the power of each coalition as a function of its members and their wealth. A coalition can take the wealth of any coalition less powerful than itself. Pillage is costless and certain, but due to the absence of a commitment technology, it can also be treacherous. Previous work on pillage games uses the concept of stable set to identify the set of allocations that can be stabilized as a balance of power (Jordan (2005) and Jordan and Obadia (2005)). Stable sets are also characterized as having a dynamic representation as a farsighted core under rational expectations (Jordan (2005, Section 6)). The balance of power in a pillage game is delicate, as evidenced by the fact that stable sets can contain at most finitely many allocations. This raises the question whether the exercise of power can be endogenously constrained to stabilize a larger set of allocations.

The more tightly power is constrained, the larger will be the set of stable allocations. However, constraints on the use of power can only be enforced by the use of power itself. An illegitimate use of power can only be discouraged by legitimizing the subsequent use of power against the transgressors. Loosely speaking, we seek the narrowest concept of legitimacy in the exercise of power that is still broad enough to be self-enforcing.

It is first necessary to extend the basic pillage game to add variables that can convey information about past actions. This information is extrinsic to the environment of the game, in the sense that it is irrelevant to the technology of power and does not enter the players' preferences, which are simply increasing in wealth. In Section 3, below, self-enforcement is represented by the static concept of the stable set of the extended pillage game. A set of allocations is called *quasi-legitimate* if it can be stabilized in the extended game. Theorem 3.7 gives a general characterization of quasi-legitimate sets. Proposition 3.6 shows that the simplest possible extension, adding only a single boolean variable, suffices to stabilize every quasi-legitimate set. Proposition 3.9 develops an iterative process for constructing the largest quasi-legitimate set. The process is based on an induction argument first used by Roth (1976) to establish the existence of the "supercore" for general abstract games. In the present setting, the process requires only finitely many steps. Section 3 also develops a more interesting extension, called the *citizenship game*. The extrinsic information identifies each player as either a citizen or an outlaw. Any act of pillage that victimizes a citizen causes all who benefit from the pillage to become outlaws. However, any pillage that victimizes only outlaws enables all players to become citizens. The citizenship game thus provides for both the punishment and redemption of outlaws. This concept of legitimacy can fail to be self-enforcing if a player can, through pillage, become wealthy enough to be too powerful to be punished. Accordingly, Proposition 3.15 shows that the set of all allocations in which only citizens possess wealth is a stable set of the citizenship game if the power function gives at least enough weight to coalitional size that no one player, even if possessed of the total wealth in the game, can be as powerful as the coalition of everyone else. This restriction on the power function, called the *no-tyranny* condition, is necessary as well as sufficient. Under the no-tyranny condition, the entire set of allocations is quasi-legitimate, since they are stable if all players are designated as citizens. The no-tyranny condition is obviously violated by the *wealth-is-power* (WIP) game, in which the power of each coalition is determined by its total wealth alone. Proposition 3.13 shows that for the WIP game, legitimacy is a fruitless concept, since no extension is capable of stabilizing any set of allocations other than the unique stable set of the basic game.

The stable set solution concept has a dynamic interpretation but a static definition. Harsanyi (1974) observes that in general, the dynamic interpretation can fail to be supported if players' expectations are explicitly taken into account. As mentioned above, stable sets of the basic pillage game are not vulnerable to this criticism, but the same cannot be said in general for stable sets of extended pillage games. Accordingly, Section 4, below, strengthens the concept of legitimacy by using the more demanding concept of a farsighted core under rational expectations. A set of allocations that can be stabilized in this fashion is called *legitimate*. Proposition 4.3 shows that quasi-legitimacy is a necessary condition for legitimacy. Theorem 4.5 shows that the citizenship game can be strengthened in such a way that the set of all allocations in which only citizens possess wealth can be stabilized as a farsighted core. Unfortunately, quasi-legitimacy is not a sufficient condition for legitimacy. Proposition 4.6 shows that no legitimate set exists for the pillage game in which the power of each coalition is the product of its total wealth and the number of its members, despite the fact that this game admits a large quasi-legitimate set.

2. Pillage games

This section defines pillage games and records some results from Jordan (2005) that will be used below. Three specific examples of pillage games are described at the end of this section.

2.1 Definitions: The set of players is the finite set $I = \{1, \ldots, n\}$, where $n \geq 2$. Subsets of I will be called *coalitions*. Let $\mathcal{C}$ denote the set of coalitions. The set of *allocations* is the set $A = \{w \in \mathbb{R}^I : w_i \geq 0 \text{ for each } i, \text{and } \sum_i w_i = 1\}$. A *power function* is a function $\pi : \mathcal{C} \times A \to \mathbb{R}$ satisfying

p.1) if $C \subset C'$ then $\pi(C', w) \geq \pi(C, w)$ for all w;

p.2) if $w'_i \geq w_i$ for all $i \in C$ then $\pi(C, w') \geq \pi(C, w)$; and

p.3) if $C \neq \emptyset$ and $w'_i > w_i$ for all $i \in C$ then $\pi(C, w') > \pi(C, w)$.

An allocation w' *dominates* an allocation w if

D) $$\pi(W, w) > \pi(L, w),$$

where $W = \{i : w'_i > w_i\}$ and $L = \{i : w'_i < w_i\}$. Domination is a binary relation on A, and will be denoted $\succ$. Domination is asymmetric but typically not transitive or even acyclic.

The property (p.3) implies that power cannot be completely independent of wealth, so the power of each coalition is endogenous. This prevents a pillage game from having a characteristic function. However, the pair $(A, \succ)$ is a special case of an *abstract game*, so the concepts of core and stable set can be defined as in Lucas (1992).

2.2 Definitions: For any set of allocations E, let $U(E)$ denote the set of allocations undominated by E, that is, $U(E) = \{w \in A : \text{no } w' \in E \text{ dominates } w\}$. The *core* of a pillage game is the set of undominated allocations, that is, the core is $U(A)$. A set of allocations E is *internally stable* if no allocation in E is dominated by an allocation in E, that is,

IS) $E \subset U(E)$.

A set of allocations E is *externally stable* if every allocation not in E is dominated by some allocation in E, that is,

ES) $U(E) \subset E$.

A set of allocations E is *stable* if it is both internally and externally stable, that is

S) $E = U(E)$.

The core always exists, but can be empty. Since $U(\emptyset) = A$, external stability (ES) implies that stable sets cannot be empty, but they can fail to exist. External stability also implies that a stable set must contain the core.

The core is the set of allocations that can be defended by force. That is, each player who holds wealth must be at least as powerful as the coalition of everyone else. The allocations most likely to satisfy this demanding requirement are those that give everything to one player.

2.3 Definition: For each i, let e^i denote the allocation that gives everything to player i, that is, $e^i_i = 1$. The e^i's are called *tyrannical allocations*.

The following result establishes that the core is nonempty only if it contains one or more of the tyrannical allocations.

2.4 Proposition: The core is the set

$$\Big\{w : \{i : w_i > 0\} = \{i : \pi(\{i\}, w) \geq \pi(I \backslash \{i\}, w)\}\Big\}.$$

For each i, the core contains e^i if and only if $\pi(\{i\}, e^i) \geq \pi(I \backslash \{i\}, e^i)$. In particular, the core is empty if and only if

NT) $\pi(\{i\}, e^i) < \pi(I \backslash \{i\}, e^i)$ for all i.

Condition (NT), which is equivalent to the emptiness of the core, will be called the *no tyranny* condition. The following proposition records the analytically useful fact that internally stable sets are finite. This contrasts with characteristic function games, whose stable sets typically contain a continuum of allocations (Lucas (1992)).

2.5 Proposition: An internally stable set can contain at most finitely many allocations.

We close this section with three examples of pillage games, which will be further discussed in subsequent sections. First, suppose the dependence of power on wealth is complete, that is, the power of each coalition is simply its total wealth. It is immediate that the core of this pillage game consists of the tyrannical allocations, together with the allocations that give half the wealth to each of two players. The unique stable set consists of all allocations in which each player with positive wealth has $w_i = (\frac{1}{2})^{k_i}$ for some positive integer k_i. In particular, the stable set for the four-player game consists of all permutations of the allocations $(1,0,0,0)$, $(\frac{1}{2},\frac{1}{2},0,0)$, $(\frac{1}{2},\frac{1}{4},\frac{1}{4},0)$, $(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8})$, and $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

2.6 The wealth-is-power pillage game (Jordan (2005)): The *wealth-is-power* (WIP) game is specified by the power function

WIP) $\pi(C, w) = \Sigma_{i \in C} w_i$.

A number $0 \leq x \leq 1$ is *dyadic* if $x = 0$ or $x = 2^{-k}$ for some nonnegative integer k. An allocation w is *dyadic* if each w_i is dyadic. Let D denote the set of dyadic allocations. For each positive integer k, let $D_k = \{w : w \text{ is dyadic and for each } i, \text{ if } w_i > 0 \text{ then } w_i \geq 2^{-k}\}$. Then $D_k \subset D_{k+1}$ for each k, and $D = \cup_k D_k$. For the WIP game, D_1 is the core and D is the unique stable set.

At the opposite extreme from the WIP game is the traditional majority game, in which the power of each coalition is equal to the number of its members. Since power is independent of wealth in this case, the traditional majority game is not a pillage game. However, there is a closely related pillage game in which the dependence of power on wealth is lexicographically secondary to coalition size, that is, relative wealth only determines the relative power of two coalitions if they have the same size. This *majority pillage* game differs from the traditional majority game in a second important respect, namely that a coalition can commit pillage without having an absolute majority, as long as it is larger than the set of its victims.

2.7 The majority pillage game (Jordan and Obadia (2005)): The *majority pillage* game is specified by any power function of the form

M) $\pi(C, w) = \#C + \alpha \Sigma_{i \in C} w_i$, where $0 < \alpha < 1$.

If n is odd, let S_n be the set of allocations consisting of all player-permutations of the allocation $(\frac{2}{n+1}, \ldots, \frac{2}{n+1}, 0, \ldots, 0)$. The core of the majority pillage game is empty. If $n = 3$, then S_3 is the unique stable set. More generally, if n is odd, then S_n is the unique symmetric stable set. If n is even and $n \geq 4$, then no symmetric stable set exists.

The final example is the game in which the power of each coalition is the product of its size and wealth. The principal interest of this game is that it has no stable set. This is proved in Jordan (2005), but it also follows from the nonexistence of a legitimate set for this game, which is proved in Section 4.

2.8 The Cobb-Douglas pillage game (Jordan (2005): The *Cobb-Douglas* game is specified by the power function

CD) $\pi(C, w) = \#C\Sigma_{i \in C} w_i$.

For the Cobb-Douglas game with $n > 2$, the core is the set of tyrannical allocations, $\{e^i : i \in I\}$, but no stable set exists.

3. Quasi-legitimacy

This section extends the basic pillage game by adding a set H of extrinsic social information that can distinguish between legitimate and illegitimate uses of power. An allocation w together with social information h constitutes a social state. An act of pillage at a social state (w, h) results in a dominating allocation w', together with new social information h'. The social information plays no role in preferences or the technology of force, so the chance from w to w' must conform to the basic domination relation $w' \succ w$. The set H and the transition from h to h' are interpreted as a social norm that characterizes a particular extension. A given pillage game can have many different extensions, including the trivial extension, which adds no information, or simply ignores the set H.

3.1 Social extensions: A *social extension* of a pillage game π consists of a set H of *social information*, and a *recording function* $\sigma : A \times H \times A \to H$. Define *social domination*, $\succ_s$, by

S) $(w', h') \succ_s (w, h)$ if $w' \succ w$ and $h' = \sigma(w, h; w')$.

A social extension will be denoted by the pair $(A \times H, \succ_s)$, and elements (w, h) of $A \times H$ will be called *states*.

The definition of a social extension embodies an implicit assumption that social information is public and the recording function σ is known to all players. The recording function is interpreted as a way of recording some information about the fact that, at the state (w, h), an act of pillage changed the allocation to w'.

A social extension $(A \times H, \succ_s)$ gives rise to stable sets of social states, that is, sets $S \subset A \times H$ that are internally and externally stable under the relation $\succ_s$. Moreover, different extensions may stabilize different sets of allocations.

3.2 Definition: Given a pillage game π, a set $X \subset A$ is *quasi-legitimate* if there is a social extension $(A \times H, \succ_s)$ of π with a stable set $S \subset A \times H$ with $X = \{w : (w, h) \in S$ for some $h \in H\}$.

The definition of a social extension places no limits on the size of H or the amount of information that can be recorded. However, the smallest nontrivial set, $H = \{0, 1\}$, suffices to stabilize every quasi-legitimate set of allocations. This is a consequence of both the mathematical elegance and conceptual weakness of the stable set concept.

3.3 The Boolean extension: Given a pillage game π and a set $E \subset A$, let $H = \{0,1\}$ and define $\sigma : A \times H \times A \to H$ as follows:

$$\sigma(w,h;w') = \begin{cases} 1 & \text{if } w' \succ w,\ w' \in E, \text{ and } w \notin E; \\ \neg h & \text{if } w' \succ w \text{ and either } w \in E \text{ or } w' \notin E; \\ h & \text{if } w' \not\succ w, \end{cases}$$

where $\neg h$ denotes the element of H not equal to h.

The Boolean extension is defined for general sets of allocations E, but is only of interest when E can be stabilized. This requires a condition called *self-protection*, which is a generalization of stability. Suppose an element $w \in E$ is dominated by an allocation w'. If an allocation $w'' \in E$ dominates w', then w'' can be very loosely interpreted as protecting w.

3.4 Definition: A set $E \subset A$ is *self-protected* if for each $w \in E$ and each $w' \in A$ such that $w' \succ w$, there exists some $w'' \in E$ such that $w'' \succ w'$. Equivalently, E is self-protected is $E \subset U^2(E)$.

Self-protection does not imply either internal or external stability, but is very useful when combined with either of them. The following proposition, which is an immediate consequence of the definitions, records the two properties that result.

3.5 Proposition: A set $E \subset A$ is externally stable and self-protected if and only if

ESSP) $$U(E) \subset E \subset U^2(E).$$

A set $E \subset A$ is internally stable and self-protected if and only if

ISSP) $$E \subset U^2(E) \subset U(E)$$

We can now show that quasi-legitimacy is completely characterized by by external stability and self-protection. We first demonstrate the sufficiency of (ESSP) using the Boolean extension.

3.6 Proposition: Given a pillage game π, let $E \subset A$ and let $A \times H, \succ_s)$ denote the Boolean extension. Define $S = \big(U(E) \times \{0,1\}\big) \cup \big((E \backslash U(E) \times \{0\}\big)$. If E is externally stable and self-protected, then S is stable, and thus E is quasi-legitimate.

Proof: We first show that S is internally stable. Let $(w,h),(w',h') \in S$. Suppose by way of contradiction that $(w',h') \succ_s (w,h)$. Then $w' \succ w$ and $w,w' \in E$. Since E is self-protected, there is some $w'' \in E$ such that $w'' \succ w'$. Therefore $w' \notin U(E)$. Hence $w,w' \in E \backslash U(E)$, so $h = h' = 1$. By the definition of $\succ_s$, this contradicts $(w',h') \succ_s (w,h)$.

We now show that S is externally stable. Let $(w,h) \notin S$. Then $w \notin U(E)$. Therefore there is some $w' \in E$ with $w' \succ w$. If $w \notin E$ then $(w',1) \succ_s (w,h)$. If $w \in E$, then

$w \in E \backslash U(E)$, so $h = 0$, which also implies that $(w', 1) \succ_s (w, h)$. Since $(w', 1) \in S$, this proves that S is externally stable.

We now complete the characterization of quasi-legitimacy by showing the necessity of (ESSP).

3.7 Theorem: A set $E \subset A$ is quasi-legitimate if and only if E is externally stable and self-protected.

Proof: Sufficiency is given by Proposition 3.6, so it remains to show necessity. Let $(A \times H, \succ_s)$ be a social extension with a stable set S satisfying $E = \{w : (w, h) \in S$ for some $h \in H\}$. We first show that E is externally stable. Let $w \notin E$ and let $h \in H$. Then $(w, h) \notin S$, so there is some $(w', h') \in S$ with $(w', h') \succ_s (w.h)$. Then $w' \in E$ and $w' \succ w$, so E is externally stable. To show that E is self-protected, let $w \in E$, $w' \in A$ with $w' \succ w$. Since $w \in E$, there is some $h \in H$ with $(w, h) \in S$. Since $w' \succ w$ there is some h' with $(w', h') \succ_s (w, h)$. Since S is internally stable, $(w', h') \notin S$. Since S is externally stable, there is some $(w'', h'') \in S$ satisfying $(w'', h'') \succ_s (w', h')$. Then $w'' \in E$ and $w'' \succ w'$, so E is self-protected.

Since stable sets necessarily satisfy (ESSP), the following corollary is immediate. Since stable sets also satisfy (ESSP), the following corollary is immediate.

3.8 Corollary: Every stable set is quasi-legitimate.

A pillage game may have many quasi-legitimate sets, but there is always a unique largest quasi-legitimate set. Moreover, an inductive argument originally used by Roth (1976) for general abstract games (see also Asilis and Kahn (1992) for further discussion and applications) gives a procedure for constructing it. For general abstract games, the process can involve transfinite induction, but the fact that internally stable sets in pillage games are always finite (Proposition 2.5) implies that the process described below stops in a finite number of steps.

3.9 Proposition: Given a pillage game π, let $G_0 = \emptyset$ and let $E_0 = A$. For each integer $t > 0$, let $G_t = U^2(G_{t-1})$ and $E_t = U^2(E_{t-1})$. Then for each $t \geq 0$, $G_t \subset G_{t+1} \subset E_{t+1} \subset E_t$. Also, for each t, $E_t = U(G_t)$ and $G_{t+1} = U(E_t)$. Moreover, there exists $T > 0$ such that $G_t = G_T$ and $E_t = E_T$ for all $t > T$. In particular,

$$G_t \subset G_T \subset E_T \subset E_t \text{ for all } t.$$

Moreover, G_T is internally stable and self-protected, and E_T is externally stable and self-protected.

Proof: Note that for any sets $X, X' \subset A$, if $X \subset X'$ then $U(X') \subset U(X)$, so $U^2(X) \subset U^2(X')$. Since $G_0 = \emptyset$, $G_0 \subset U^2(G_0) = G_1$. Hence by iteration, $G_t \subset G_{t+1}$ for all t. Since $G_0 = \emptyset$, G_0 is internally stable and self-protected. Since the (ISSP) inclusions are preserved by $U^2(\cdot)$, it follows that G_t is internally stable and self-protected for all t. Let $G^* = \cup_t G_t$. Since each G_t is internally stable and the sequence of sets is increasing, it follows that G^* is internally stable. By Proposition 2.5, all internally stable sets are finite, so G^* is finite, which implies that $G^* = G_T$ for some integer T.

Since $E_0 = A$, $E_1 = U^2(E_0) \subset E_0$. Since $U^2(\cdot)$ preserves inclusion, the $E_{t+1} \subset E_t$ for all t. Also, since $E_0 = A$, E_0 is externally stable, that is, $U(E_0) \subset E_0$. Since $U^2(\cdot)$ preserves this inclusion, E_t is externally stable for all t. Since $\emptyset = G_0 \subset E_0 = A$, applying the $U^2(\cdot)$ operator successively to this inclusion shows that $G_t \subset E_t$ for all t. Note that $E_0 = U(G_0)$. Hence $U(E_0) = U^2(G_0) = G_1$ and $E_1 = U^2(E_0) = U(G_1)$. Hence $E_t = U(G_t)$ for all t, so the required E_T is $E_T = U(G_T)$. Finally, since $E_T = E_{T+1} = U^2(E_T)$, E_T is self-protected.

3.10 Theorem: Given a pillage game π, let $E \subset A$ be quasi-legitimate. Then $G_T \subset E \subset E_T$. In particular, E_T is the largest quasi-legitimate set.

Proof: Since E is quasi-legitimate, Theorem 3.10 implies that E satisfies (ESSP). We will prove the result by induction on t. Suppose that for some t, $G_t \subset E \subset E_t$. Since E is self-protected, $E \subset U^2(E)$. Since $E \subset E_t$, $U^2(E) \subset U^2(E_t) = E_{t+1}$, so $E \subset E_{t+1}$. Since E is externally stable and $E \subset E_t$, $G_{t+1} = U(E_t) \subset E$. Thus, if $G_t \subset E \subset E_t$, then $G_{t+1} \subset E \subset E_{t+1}$. Since $G_0 = \emptyset$ and $E_0 = A$, $G_0 \subset E \subset E_0$, it follows by induction that $G_T \subset E \subset E_T$. The final assertion follows from the fact that E_T satisfies (ESSP).

The following useful corollary is immediate.

3.11 Corollary: If $G_T = E_T$, then E_T is the unique quasi-legitimate set and also the unique stable set.

3.12 Corollary: If π satisfies the no tyranny condition (NT), then A is quasi-legitimate.

Proof: By Proposition 2.4, (NT) implies that the core, $U(A)$, is empty, so the iterative process terminates at the first step, with $E_T = E_0 = A$.

The iterative process for constructing E_T can be applied to derive E_T for the three specific pillage games defined in the preceding section. The majority pillage game with at least three players satisfies the no tyranny condition, so the entire set of allocations is quasi-legitimate. For the Cobb-Douglas pillage game, the iteration terminates after the first step, yielding a large quasi-legitimate step, despite the absence of a stable set for this game. The WIP game, in contrast, allows no room for quasi-legitimacy beyond the unique stable set.

3.13 Proposition:

a) For the WIP game $G_T = E_T = D$.

b) For the majority game with $n > 2$, $E_T = A$.

c) For the CD game,

$$E_T = \{e^i : i \in I\} \cup \{w : w_i \leq \frac{p(w) - 1}{p(w)} \text{ for all } i\},$$

where $p(w) = \#\{i : w_i > 0\}$.

Proof: The proof given in (Jordan (2005)) that D is the unique stable set for the WIP game uses an iterative construction that can be slightly modified to show (a). The majority pillage game with $n > 2$ is a special case of Corollary 3.12. For the CD game, if $n = 2$, it is immediate that $G_1 = E_1 = \{(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2})\}$. If $n \geq 3$, the core is $G_1 = U(A) = \{e^i : i \in I\}$. An allocation w is undominated by e^i if and only if $w_i \leq \frac{p(w)-1}{p(w)}$, so $E_1 = U(G_1) = G_1 \cup \{w : w_i \leq \frac{p(w)-1}{p(w)}$ for all $i\}$. Now suppose that $n = 3$. Then $G_2 = U(E_1) = G_1 \cup \{w : w_i = \frac{2}{3}$ and $w_j = w_k = \frac{1}{6}$ for some $i, j, k\}$, and $E_2 = U(G_2) = E_1$, so $E_T = E_1$. If $n > 3$, then $G_2 = U(E_1) = G_1$, so again $E_T = E_1$.

The abstract simplicity of the Boolean extension is helpful in characterizing quasi-legitimacy, but yields little insight about how power can be constrained. The following more explicit, albeit less general extension is much easier to interpret. Suppose that each player can be designated as either a *citizen* or an *outlaw*. A social state then consists of an allocation together with a given designation for each player. The player designations change with pillage in the following way. If any citizen is pillaged, all players who benefit from the pillage become outlaws. However, if only outlaws are pillaged, then all players become citizens. This social norm protects the property of citizens, provided that the no tyranny condition is satisfied. Any player who participates in the pillage of a citizen becomes an outlaw, and is thus left open to pillage by anyone. Even under the no tyranny condition, it may not be possible for the remaining citizens to pillage all of the outlaws. For example, in the majority pillage game, suppose that player 1 is a citizen and is pillaged by the coalition of everyone else. Then player 1 will need to enlist the aid of at least half of the outlaws to get some of his property back. In this case, half of the outlaws profit a second time by betraying their fellow outlaws. The fact that at least one outlaw can be left with nothing after the second pillage is what discourages the original pillage. This is established using the static stable set concept, and further strengthened in the next section by making the dynamics explicit.

3.14 The citizenship extension: Let $H = \mathcal{C}$ and define $\sigma : A \times H \times A \to H$ as follows:

$$\sigma(w, C; w') = \begin{cases} I \backslash W & \text{if } L \cap C \neq \emptyset; \\ I & \text{if } L \neq \emptyset \text{ and } L \cap C = \emptyset; \\ C & \text{if } L = \emptyset, \end{cases}$$

where $L = \{i : w'_i < w_i\}$ and $W = \{i : w'_i > w_i\}$.

3.15 Proposition: Let $(A \times \mathcal{C}, \succ_s)$ be the citizenship extension of a pillage game π. Let $S = \big\{(w, C) : \{i : w_i > 0\} \subset C\big\}$. The S is a stable set if and only if π satisfies the no tyranny condition (NT).

Proof: First assume that π has an empty core. Let $(w, C) \in S$ and let (w', C') be a social state that dominates (w, C). Let $W = \{i : w'_i > w_i\}$ and $L = \{i : w'_i < w_i\}$. Since $(w, C) \in S$, $L \subset C$ so $C' \cap W = \emptyset$. Then for each $i \in W$, $w'_i > 0$ and $i \notin C'$, so $(w', C') \notin S$. Thus S satisfies internal stability. Let $(w, C) \notin S$. Then $w_i > 0$ for some $i \notin C$. Let $C' = I \backslash \{i\}$ and let w' satisfy $w'_i = 0$ and $w'_j > w_j$ for each $j \neq i$. Then $(w', C') \in S$. By (NT), $\pi(C', e^i) > \pi(\{i\}, e^i)$, so $w' \succ w$. Hence $(w', C') \succ_s (w, C)$, so S satisfies external stability.

Now suppose that π does not satisfy (NT), so $\pi(I \backslash \{i\}, e^i) \leq \pi(\{i\}, e^i)$ for some i. Then e^i is undominated, so the social state $(e^i, I \backslash \{i\})$ is undominated. Since $(e^i, I \backslash \{i\}) \notin S$, it follows that S does not satisfy external stability.

4. Legitimacy

Quasi-legitimacy is analytically convenient but conceptually inadequate. Quasi-legitimate sets have a simple and complete characterization, and there is a finite iterative procedure that generates the largest quasi-legitimate set. Unfortunately a stable set of a social extension, unlike a stable set of the basic pillage game, is vulnerable to Harsanyi's criticism of the stable set concept (Harsanyi (1974)). To put Harsanyi's criticism in the present context, let $(A \times H, \succ_s)$ be a social extension, and let $S \subset A \times H$ be a stable set. The set S is interpreted as self-enforcing based in part on the following argument. If $(w, h) \in S$ is dominated by some (w', h'), then since S is internally stable, $(w', h') \notin S$. Since S is externally stable, there is some $(w'', h'') \in S$ that dominates (w', h'). Since S is internally stable (w'', h'') does not dominate (w, h), so the prospect of moving to (w'', h'') discourages the move to (w', h'). Harsanyi observed that if the players who benefit from the move to (w', h') benefit still further, or at least don't lose anything in the subsequent move to (w'', h''), they will happily force the move to (w', h') in order to achieve in two moves what internal stability prevents them from achieving in one. Of course, the natural way to resolve this confusion is simply to make the expectation of the subsequent move from (w', h') to (w'', h'') explicit for all players. For the basic pillage game, in which social information is absent, Jordan (2005, Section 6) shows that stable sets not only survive the Harsanyi criticism but can be characterized as the only sets that do. Unfortunately, the stable sets of social extensions fare much worse.

The vulnerability of the Boolean extension to the Harsanyi critique is pervasive and in some cases irreparable. The largest quasi-legitimate set for Cobb-Douglas pillage game is very large, as described by Proposition 3.13, but Proposition 4.6 below shows that the Harsanyi critique prevents the existence of any legitimate set.

The citizenship game, as described in the preceding section, has an interesting deficiency that is subject to the Harsanyi critique as well. The recording function σ uses the allocations w and w' to identify the pillaging coalition and its victims. If actions are motivated by the anticipation of subsequent actions, the allocation w' is only a step on the way to w''. The definition of σ allows farsighted players to force an action w' with the object of diverting punishment from themselves, and enabling themselves to benefit from the punishment of players they have caused to be wrongly identified as transgressors. This is corrected in the following definition, which enables the recording function to correctly identify the coalition that forced an action and the players who unsuccessfully opposed it.

4.1 Definitions: A *dynamic extension* of a pillage game π consists of a set H and a recording function $\delta : A \times H \times \mathcal{C}^2 \times A \to H$. An *expectation* is a function $f : A \times H \to A \times H$ satisfying $f^2 = f$. Since $f^2 = f$, the set $f(A \times H)$ is the set of *stationary states.* Given an expectation f, and states (w, h) and (w', h'), let $(w'', h'') = f(w', h')$. The state (w', h') dominates (w, h) in expectation if

i) w'' dominates w; and

ii) $\delta(w, h, W, L, w') = h'$,

where $W = \{i : w''_i > w_i\}$ and $L = \{i : w''_i < w_i\}$.

The concept of domination in expectation is only of interest when the expectations are fulfilled. The following definition of *consistent expectations* provides the requisite rational expectation condition.

4.2 Definitions: An expectation f is *consistent* if for each (w, h), either

i) $f(w, h)$ dominates (w, h) in expectation; or

ii) (w, h) is undominated in expectation and $f(w, h) = (w, h)$.

The *farsighted core* is the set of states that are undominated in expectation. If S is a farsighted core for a consistent expectation, the set $\{w : (w, h) \in S \text{ for some } h \in H\}$ is said to be *legitimate.*

The following result shows that quasi-legitimacy is a necessary condition for legitimacy.

4.3 Proposition: Every legitimate set is also quasi-legitimate.

Proof: Let $X \subset A$ be legitimate, and let S be a farsighted core for a consistent expectation f, with $X = \{w : (w, h) \in S \text{ for some } h \in H\}$. By Theorem 3.7, it suffices to show that X is externally stable and self-protected. Let $w \notin X$ and let $h \in H$. Then $(w, h) \notin S$. Then by 4.2(i), $f(w, h) = (w', h')$ for some (w', h') that dominates (w, h) in expectation. Since

$f^2 = f$, $(w', h') \in S$, so $w' \in X$ and w' dominates w, which proves that X is externally stable. Let $w \in X$ and let w' dominate w. If $w' \notin X$ then since X is externally stable, there is some $w'' \in X$ that dominates w'. Suppose $w' \in X$. Let $W = \{i : w'_i > w_i\}$, $L = \{i : w'_i < w_i\}$, and $h' = \delta(w, h, W, L, w')$. Then $(w', h') \notin S$, otherwise (w', h') would dominate (w, h) in expectation. Therefore, as above, there is some $w'' \in X$ that dominates w'.

The following definition adapts the recording function of the citizenship game as required for a dynamic extension. The resulting version of the citizenship game is then shown to stabilize the full set of allocations as a farsighted core, provided, as before, that the no tyranny condition is satisfied.

4.4 Definition: The *dynamic citizenship extension* is defined as follows. Let $H = \mathcal{C}$ and define $\delta : A \times H \times \mathcal{C}^2 \times A \to H$ by

$$\delta(w, C, W, L, w') = \begin{cases} I \backslash W & \text{if } L \cap C \neq \emptyset; \\ I & \text{if } L \neq \emptyset \text{ and } L \cap C = \emptyset; \\ C & \text{if } L = \emptyset. \end{cases}$$

Let $S = \big\{(w, C) : \{i : W_i > 0\} \subset C\big\}$.

4.5 Theorem: Let π be a pillage game that satisfies the no tyranny condition (NT). For the dynamic citizenship extension, let $S = \big\{(w, C) : \{i : W_i > 0\} \subset C\big\}$, and define the expectation $f : A \times H \to A \times H$ by

$$f(w, C) = \begin{cases} (w, C) & \text{if } (w, C) \in S; \\ (w', I) & \text{for some } w' \text{ that dominates } w \text{ and satisfies} \\ & \{i : w'_i < w_i\} \subset I \backslash C, \text{ and } w'_i = 0 \text{ for some } i \in I \backslash C, \\ & \text{if } (w, C) \notin S. \end{cases}$$

Then f is consistent and S is the farsighted core. In particular, the entire set A is legitimate.

Proof: We first show that f is a well-defined expectation. By the definition of S, $(w, I) \in S$ for all w, so $f^2 = f$. Second, let $(w, C) \notin S$. We need to show that the allocation w' required in the definition of f exists. Since $(w, C) \notin S$, there is some i^o with $w_{i^o} > 0$ and $i^o \notin C$. Let $w' = (w_1 + \frac{w_{i^o}}{n}, \ldots, w_{i^o-1} + \frac{w_{i^o}}{n}, 0, w_{i^o+1} + \frac{w_{i^o}}{n}, \ldots, w_n + \frac{w_{i^o}}{n})$. Since π satisfies (NT), w' dominates w, and w' clearly satisfies the other required properties as well. Therefore f is a well-defined expectation.

We now show that states in S are undominated in expectation. Let $(w, C) \in S$ and $(w', C') \in A \times \mathcal{C}$. First suppose that $(w', C') \in S$. Then (w', C') cannot dominate (w, C) in expectation, since (w', C') is stationary, and since $(w, C) \in S$, w' dominates w and $\{i : w'_i < w_i\} \subset C$, so by the definition of δ, $(w', C') \notin S$. Now suppose $(w', C') \notin S$ and suppose by way of contradiction that (w', C') dominates (w, C) in expectation. Since $(w', C') \notin S$, there is some $(w'', C'') \in S$ with $f(w', C')$ dominates (w', C') in expectation and $f(w', C') \in S$. Let $(w'', C'') = f(w', C')$. Since (w', C') dominates (w, C) in expectation, $C' = \delta(w, C, W, L, w')$,where $W = \{i : w''_i > w_i\}$ and $L = \{i : w''_i < w_i\}$. Since

$(w, C) \in S$, $L \cap C \neq \emptyset$, so by the definition of δ, $I \backslash W = C'$. Then , by the definition of f, there is some $i \in W$ with $w_i'' = 0$,which contradicts the definition of W, which proves that all states in S are undominated in expectation.

Let $(w, C) \notin S$. We will complete the proof by showing that $f(w, C)$ dominates (w, C) in expectation. Let $(w', I) = f(w, C)$. Then (w', I) is stationary and w' dominates w. Let $W = \{i : w_i' > w_i\}$ and $L = \{i : w_i' < w_i\}$. Then $I = \delta(w, C, W, L, w')$, so (w', I) dominates (w, C) in expectation. This proves that f is consistent and that S is the set of states that are undominated in expectation.

The following result shows that the Cobb-Douglas pillage game with more than two players has no legitimate sets. This result also implies that quasi-legitimacy is not sufficient for legitimacy.

4.6 Proposition: For the CD game with $n > 2$, no legitimate set exists.

Proof: We will show there does not exist a consistent expectation for the Cobb-Douglas game with more than two players. Let $w \in E_T$ satisfy

$$(*) \qquad w_1 \geq w_2 \geq \cdots \geq w_n > 0,\ w_1 = \frac{n-1}{n}, \text{and } \frac{1}{n} > w_2 > \frac{n-2}{n(n-1)}.$$

Then $\pi(\{1\}, w) = \pi(\{2, \ldots n\}, w)$ and $\pi(\{2\}, w) > \pi(\{3, \ldots, n\}, w)$. Let $w' \in E_T$ also satisfy $(*)$ with $w_2' > w_2$ and $\Sigma_{i>2} w_i' < w_n$. Let $W = \{2\}$ and $L = \{3, \ldots, n\}$. Suppose by way of contradiction that f is a consistent expectation for some dynamic extension. Let $h \in H$, and let $h' = \delta(w, h, W, L, w')$. We will first show that (w', h') dominates (w, h) in expectation. If $f(w', h') = (w', h')$, this is immediate, since $w' \succ w$ and $h' = \delta(w, h, W, L, w')$. Suppose instead that $f(w', h') = (w'', h'') \neq (w', h')$. Then since $f^2 = f$, $f(w'', h'') = (w'', h'')$, so (w'', h'') is an element of the farsighted core and $w'' \succ w'$. By Proposition 4.3, $w'' \in E_T$, which, together with the fact that $w'' \succ w'$, implies that w'' also satisfies $(*)$, with $w_2'' \geq w_2' > w_2$ and $w_i'' < \Sigma_{i>2} w_i' < w_i$ for all $i > 2$. Since $h' = \delta(w, h, W, L, w')$, this again shows that (w', h') dominates (w, h) is expectation. If $(w', h') = f(w', h')$ then since f is consistent, (w', h') is undominated in expectation. However, since w' satisfies $(*)$, the above argument can be repeated to construct a state (w''', h'') that dominates (w', h') in expectation. Therefore $(w'', h'') = f(w', h') \neq (w', h')$ and (w'', h'') is in the farsighted core. However, w'' also satisfies $(*)$, so (w'', h'') is also dominated in expectation, which proves that f is not a consistent expectation.

A stable set of a pillage game is shown to be a farsighted core in Jordan (2005, Section 6). The trivial dynamic extension, $H = \{0\}$, extends this result to show that every stable set of a pillage game is legitimate. For this reason, the nonexistence of a legitimate set in the Cobb-Douglas game also implies the nonexistence of a stable set.

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