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Asset return dynamics and the FX risk premium in a decentralized dealer market

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Abstract

The paper proposes a continuous time model of an FX market organized as a multiple dealership. The dealers have costly access to best available quotes. They interpret signals from the joint dealer–customer order flow and decide upon their own quotes and trades in the inter-dealer market. Each dealer uses the observed order flow to improve the subjective estimates of relevant aggregate variables, which are the sources of uncertainty. The risk factors are returns on domestic and foreign assets and the size of the cross-border dealer transactions in the FX market. These uncertainties have diffusion form and are dealt with according to the principles of portfolio optimization in continuous time. The model is used to explain the country, or risk, premium in the uncovered national return parity equation for the exchange rate. The two country premium terms that I identify in excess of the usual covariance term (consequence of the "Jensen inequality effect") are: the dealer heterogeneity-induced inter-dealer market order flow component and the dealer Bayesian learning component. As a result, an "order flow-adjusted total return parity" formula links the excess FX return to both the "fundamental" factors represented by the differential of the national asset returns, and the microstructural factors represented by heterogeneous dealer knowledge of the aggregate order flow and the fundamentals.

JEL classification: F31; G11; G29; D49; D82

Keywords: FX microstructure; Optimizing dealer; Diffusion uncertainty; Uncovered parity

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1. Introduction[1]

The paper develops a continuous time model of the currency pricing in an inter-dealer FX market and derives consequences for the risk premium structure in the formula for uncovered parity of national asset returns. It is an asset pricing model with an additionally imposed decentralized dealer market organization. The economy defined in the paper is closer to the reality of FX and many other security markets than the standard asset pricing paradigm. A continuous time, diffusion uncertainty set-up has been chosen for its better analytical tractability compared to discrete time models. I demonstrate that the forex microstructure has an impact on the dynamics of the risk premium, by linking the behavior of the latter to

(1) the order flow received by dealers–domestic residents from other market users,
(2) errors in the dealers' assessments of the aggregate cross-border order flow and economic fundamentals.

The model is a formalized tool for the study of general mechanisms jointly driving the FX return and other asset returns in a decentralized dealership market, with the stress on the effects that cannot be captured by a representative agent Walrasian market approach. The methodological message is twofold.

First, the model demonstrates that in a decentralized dealer market, the rate of return on every asset satisfies a consumption-based CAPM formula adjusted for the statistics of the aggregate order flow between the selling and the purchasing part of the market users (dealers and non-dealers). If, in addition, the market makers do not know the parameters of the risk processes exactly, the CCAPM-correcting term contains the subjective estimation error of the said aggregate order flow. This, with minor modifications, is applicable to any asset market made simultaneously by multiple dealers outside an organized exchange, such as many bond markets.

Second, the model delivers a specific result for the FX market with the same organization. It shows that the FX return and returns on any two assets denominated in different currencies satisfy an uncovered parity condition in which the FX risk premium depends on the statistics of the aggregate order flow between the selling and the purchasing part of the market users. If the market makers do not know the parameters of the risk processes exactly, the risk premium contains the subjective estimation error of the aggregate order flow. Since for internationally traded currency pairs, the forex has the character of a decentralized dealership, with some, but not all, trades executed through brokers, this result is applicable quite generally.

This paper focuses on the FX market, for which the existence of the above-mentioned generalized uncovered parity theorem allows one to reduce the number of unobservable variables in the pricing formulae and get closer to empirically verifiable results. Namely, it is possible to express the price of foreign currency in this model, as of any other asset, through the marginal utilities (analogously to the standard

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International consumption-based CAPM by Lucas (1982), where the exchange rate is equal to the marginal rate of substitution between domestic and foreign consumption). By combining this with the similar pricing equations for one domestic and one foreign asset, one arrives at the *relative currency pricing* by means of the chosen asset pair. The differential form of this relationship is the said generalized uncovered parity. In the context of this no-arbitrage condition that prevails in a decentralized dealer market, I then study the consequences of the FX dealer heterogeneity, imperfectly observed risk distributions and the resulting non-trivial inter-dealer order flow, for the excess FX return over the cross-border asset return differential.

The information set of each dealer is generated by the observed asset return processes and the privately received order flow. The latter provides an imprecise signal of the aggregate order flow. In the imperfect information version of the model, the expected rates of return on assets (drifts of the cumulative return processes) are unobservable. This is a conventional feature of the literature on asset pricing under partial observations (Detemple, 1986; Dothan and Feldman, 1986; Genotte, 1986; Brennan, 1998; Zapatero, 1998). Unobservable parameters of the risk processes do not come up so often in microstructure finance, because there it is usual to work directly with the uncertain final realization of the asset value (see later). I do not operate this unnatural shortcut in the chosen dynamic asset pricing context. The microstructure theory has suggested the way of thinking about the market mechanism to be modeled, while stochastic finance has provided the techniques to be used. Therefore, most technical assumptions are standard in the continuous time asset pricing theory, but may be new in the microstructure finance context.

The model allows one to address two types of FX market effects. The first are the "long-run" properties of the exchange rate, such as a (generalized) uncovered parity or competitive quoting by multiple dealers around a clearing price. The second group, which is "real time" by nature, refers to information extraction procedures and Bayesian belief updating by participants in the inter-dealer market in the face of changing fundamentals. Combining the said objectives, I derive an "order flow-adjusted" uncovered parity of national asset returns with respect to the exchange rate return, with the country premium depending on the order flow from non-resident market users to resident dealers. One term in the premium is present under both perfect and imperfect information and comes from heterogeneous liquidity needs of the dealers. The other term is a consequence of an individual dealer's imperfect information about the aggregate order flow and national asset return statistics. Bayesian learning and belief updating by the dealers then leads to long-lasting shifts in the country premium.

The obtained result generalizes the uncovered parity property of the exchange rate that comes up naturally in any optimizing model of international asset pricing. (Under the Walrasian market clearing assumption, this uncovered parity would follow from the international consumption-based CAPM.) We choose to call it the uncovered total return parity (UTRP), to make a distinction from the much-compromised uncovered interest rate parity of naïve no-arbitrage models. UTRP associates the exchange rate expectations for a given period with the difference in total returns (instantaneous dividend over price plus capital gain) on a pair of representative securities. These total returns coincide with yields to maturity in continuous time. Accordingly, the return quoted in a secondary market and not the money market loan/deposit rate, which is predetermined for the time interval in question, constitutes the continuously updated measure of the expected move in the exchange rate (see Derviz, 2002, for details, including empirical verification of UTRP). This parity theorem would be valid exactly and permanently in a Walrasian auctioneer setting which ignores microstructure, with a representative agent and markets clearing at each moment. However, when one studies the exchange rate formation in a dealership market, deviations from the fundamental UTRP come about as a natural consequence of agent heterogeneity. These deviations reflect information and inventory flows between dealers and investors. The proposed model establishes a link between an individually observed order flow, Bayesian filtering of imprecisely known fundamentals by the dealer, and the seemingly unwarranted variability of the country premium in the UTRP formula. The model predicts long-lived deviations from UTRP due to the order flow effects. They are always present when dealers are learning new information from the received orders. During such periods:

1. the exchange rate and the inter-dealer order flow variability must be higher than under complete symmetric information;
2. dynamic processing of new signals by the dealers leads to deviations from UTRP that persist until the new information has been absorbed; under certain circumstances, Bayesian learning by the dealers can lead to a permanent revision of the country premium level in the uncovered parity relation.

My model, as opposed to most others in FX microstructure, maintains a firm connection to the intertemporal optimization foundations of dynamic asset pricing. For example, Evans and Lyons (2002), in their reduced form result, link the *realized FX-payoff*, the exchange rate and the order flow. In my model, there is no single FX-payoff, each foreign currency-denominated asset has an uncertain payoff of its own. My order flow-adjusted uncovered parity is an equilibrium-determined function of the joint distribution of all existing market uncertainties. On the contrary, in the regressions of Evans and Lyons (2002), the already-mentioned FX-payoff is proxied by the change in the one-period interest rate differential. This is possible as an ad hoc specification, whereas a model based on first principles of portfolio optimization would usually contain a linear dependence between the returns themselves. I share the view of Evans and Lyons (2002) that the client order flow reflects the portfolio shifts of the public. In my model, orders of optimizing investors are split between dealers. The latter are heterogeneous and generate an inter-dealer order flow. Due to the decentralized dealership market structure, portfolio shifts are private information and the aggregate order flow has to be discovered by the dealers.

A frequently invoked question is the possible "frequency mismatch" between the standard international macroeconomic models of the exchange rate and the microstructure models. The latter demonstrate that the exchange rate is closely related to order flow *monitored in a particular structure* determined by the prevailing forex trading mechanism. Transaction-level frequency data are best at *revealing* this relation, whereas lower-frequency observations tend to obfuscate it (because they ignore the said order flow structure and are likely to miss the orders and time moments that contain important price signals). However, the fact itself is frequency-independent. My model shows it by representing exchange rate as a function of indirect marginal utilities, or shadow prices. The latter can be obtained by forward-integrating the Euler equations, which involves expectations of long-term fundamentals. So, this exchange rate formula has a long-horizon validity. On the other hand, the same equation contains the order flow variable best observable at transaction frequencies. Although empirically testable and macrorelevant exchange rate results come out of a number of microstructure models, including the seminal Evans and Lyons (2002), one, my model, in addition, demonstrates that the underlying cause of the relationship is present irrespective of the fluctuation measures employed in empirical analysis.

1.1. Literature review

1.1.1. Microstructure finance sources

To highlight the “right” price discovery process starting with the end-users of the foreign currency (investors) and going through the dealers who learn from them, the existing microstructure literature usually makes a conceptual distinction between clients and dealers. This happens both in single-dealer sequential trade models of Glosten and Milgrom (1985) or Easley and O’Hara (1987) with a single risk-neutral market maker, and in multiple dealer models such as Vogler (1997) or Evans and Lyons (2002). This distinction is barely possible in practice. More importantly, the real life roles of the investor and dealer in the FX market are often combined in the performance of cooperating parts within the same company. Most typically, a major commercial or investment bank has dealers who service its customers and participate in the inter-dealer market, but also operates proprietary trading desks that exercise FX transactions for the needs of its own portfolio management. Access to electronic cross-border security trade and information systems is not a priviledge of dealers either: the community of Reuters and/or Bloomberg quote screen users includes a considerable part of the internationally active companies of very versatile profiles. Access to these facilities becomes highly attractive to any company whose cross-border operations attain a certain size. Accordingly, most of available price and trade size data carry no stamp of the purpose of any given FX transaction.

One of the exceptions in the literature in terms of client–dealer separation is the paper by Madhavan and Smidt (1993), in which a dealer is also an active investor. Similarly to Madhavan and Smidt, I model a synthetic dealer–investor. All agents in the present model are market users. They are either *local*, trading only with one selected dealer, or *global*, having access to the inter-dealer market. That is, global market users are able to both search for the best dealer quotes and exercise a trade with the dealer of their choice. The only distinction between dealers and the rest of the global market users is that the former assume the market making function. Apart from that, dealers have the same intrinsic motive for holding foreign currency positions as any other market user, since they are active in the same lines of international business as the latter. The respective roles of local and global market users in the model can be roughly summarized as follows: order flow of the locals creates noise and can generate exogenous asymmetry in dealer positions; order flow of the globals acts as a signal that transmits information between dealers.[2]

The majority of information-oriented forex microstructure models define the signals as unbiased estimates of some "true" or "fundamental" value of the exchange rate. This feature was inherited from microstructure studies of other markets (e.g. equities, where this true value is associated with revealed earnings or dividend data). There are few exceptions in the literature, such as Vogler (1997) or Foucault (1999). This comes as no surprise since both papers model specific market mechanisms (the former—a direct dealership and an auction, the latter—a limit order book) in which there is no natural place for an external asset liquidation value. However, the inter-dealer market of Vogler (1997) is not a place where information is learned. It serves entirely for redistribution of received risk from client trades. We go beyond Vogler's pure risk-sharing motive for trade, since our optimizing dealers place orders in the inter-dealer market on the basis of new knowledge extracted from the current-period observations. On the other hand, the contents of this knowledge refers to expected returns of the real domestic and foreign assets and the general direction of the FX market, not the "true" future currency value. In the case of the exchange rates, no such absolute truth exists as *an exogenous parameter* of the model, only as its *solution*. In particular, although the investor allocates the purchased foreign cash optimally across available assets, the distributional parameters of the rate of return on this position are not perfectly known in advance. (This rate of return is, of course, different from the interest rate paid on foreign cash. So, the expected present value of future cash interest payments is not equal to the equilibrium exchange rate under any of the equivalent martingale measures that might exist in this market. In fact, the present set-up does not require an equivalent martingale measure to exist.) In order to create a realistic pattern of signal processing by the dealer, I let the received order flow contain information about the aggregate behavior of the trade initiators. In short, I am not modeling either pure "information" or pure "inventory" motives for trade, but rather, an "information about inventory" motive.

Another highly stylized feature of existing models of the inter-dealer market is a strict sequencing of information-laden events in them. (cf. Evans and Lyons (2002): first a round of dealer trades with customers signaling fundamental information, then an inter-dealer round to redistribute inventory risk between dealers, then another round of customer trades to unwind open positions.) On the contrary, data on FX trades of a given financial institution reflect three simultaneously evolving random processes pertaining to own investment needs, the inter-dealer and the customer–dealer operations. In other words, at each moment, the dealer observes the three named categories of transactions simultaneously and has to exercise a belief update based on these observations. Our continuous time stochastic model of the dealer behavior accommodates this feature in a convenient way.

[2] This understanding of the aggregate signaling effects of the market users' order flow can be supported by the reference to the model of Bernhardt and Hughson (1997). There, customers are splitting orders between dealers to equalize marginal benefits from trade at different pricing schedules. The implicit marginal benefits equalization is the reason why, in our model, the global market user order flow observed by one dealer contains information on the order flow received by other dealers, as well.

Since each dealer has a local customer base that trades with him exclusively, there is enough space for a non-trivial bid–ask spread in his quotes. The adverse selection reason for the existence of spreads identified in most models with informational heterogeneity of traders, namely the compensation of losses from trades with the informed by profits made in trades with the uninformed, is implicitly present in our setting as well. The role of the informed is played by the global traders, while that of the uninformed is played by the local traders. The latter, when they trade at given spreads, are subject to the exercise of monopoly power by their dealer. This understanding is close to the ideas of Copeland and Galai (1983) and Perraudin and Vitale (1996). However, the present model generates non-trivial—and variable—bid–ask spreads even when there is more than one dealer in the market. The existence of global investors extends the competition from pure inter-dealer interaction to the area of dealer–customer relations.

The existence of informational heterogeneity between investor groups of different residences is pivotal in generating non-trivial order flows and additional sources of exchange rate volatility. This type of heterogeneity has already found reflection in finance literature. For instance, Brennan and Cao (1997) argue that asymmetry between domestic and foreign investors in the knowledge of domestic asset returns has an impact on the direction and volume of cross-border equity trade. More specifically, a working paper by Seasholes (2000) argues that large foreign investors in emerging markets have an informational advantage and cash in higher returns on domestic equity, compared to the majority of domestic investors. (Therefore, cross-border informational asymmetry does not always mean an advantage for residents.) Our model shares the Brennan and Cao view that residency-based informational heterogeneity matters, but also allows for difference in informational endowments between local and global investors, in accordance with the Seasholes conjecture.

Beside macrostructure finance, this paper benefited from a number of other directions in the international macroeconomics and stochastic finance literature. A brief overview and comparisons are given below.

1.1.2. Continuous time microfinance

There already exists a line of literature that develops the findings of the Kyle (1985), risk-neutral market maker model in continuous time. For example, Back (1992), Back and Pedersen (1998) and Back et al. (2000) work with price and cumulative orders in a security market (applicable to forex) in semimartingale form, with diffusion components originating in the action of noise traders. When one considers optimizing models for risk-sensitive agents, the stochastic maximum principle becomes an even more powerful tool of analyzing the equilibrium price and order flow dynamics in continuous time, once the dealer optimization problem is properly defined in this setting. The crucial challenge here is to identify and interpret the information asymmetry and Bayesian belief updating phenomena and their role in the obtained solutions.[3]

[3] The previously mentioned continuous time models of security trade by Back and colleagues are based on Kyle's auctioneer and do not include dealers explicitly. Besides, these models abstract from the usual Brownian asset return uncertainties faced by an investor and concentrate on the market interpretation of narrowly defined exogenous signals about the asset value.

A number of authors addressed this task in a continuous time portfolio model of the Cox–Ingersoll–Ross type with heterogeneous beliefs.

1.1.3. Kalman filtering and Bayesian learning

A major contribution to the continuous time portfolio choice and asset pricing literature under imperfect information was made by Detemple (1986), Dothan and Feldman (1986) and Genotte (1986). Detemple applies the abstract stochastic control theory under imperfect information (overcome by Kalman-filtering the observations) to the consumer/investor problem with a partially observed state process and derives the "separation principle" for this problem. The latter says that the agent first forms expectations of the unobserved states conditional on accessible information and then calculates the optimum by using the estimates instead of the true values. Dothan and Feldman (1986) build on this separation principle to study general equilibrium asset pricing in an economy, which is an extension of the Cox–Ingersoll–Ross one to the case of unobservable asset return drifts. Genotte (1986) uses the same framework to investigate the impact of Bayesian inference on the properties of portfolio rules.

A decade later, Brennan and colleagues (one representative paper in this line of literature is Brennan, 1998) investigated the effects of learning on risky asset hedging demands in the same continuous time Kalman–Bucy framework. The analysis is conducted against the standard Merton representative investor benchmark. Learning about the expected return does not rely on observations other than the risky return itself. That is, the belief update process is degenerate compared to Detemple (1986) or Genotte (1986) where investors learn from other signals as well. The obtained results do not address the impact of learning on asset price formation or evolution.

All the said papers work with a representative investor in a frictionless (except for imperfect observations) asset market, their source of informational imperfection and estimation errors is not specified, and all uncertainties are the same for everyone. No market structure or trading process is involved in the analysis.

Zapatero (1998) uses the formalism developed in Detemple (1986) to analyze asset price volatility under heterogeneous investor beliefs about the same state process (as opposed to imperfect observations by a representative agent studied in earlier literature). We exploit the general idea of this paper that different types of agents may *observe* the same process, but *differently interpret* it in terms of the assigned statistical properties. Zapatero (1998) analyzes the interaction outcome of two Merton-type investors (an "optimist" and a "pessimist") who apply different parametrizations to the same observation process. This has consequences for the equilibrium price. In our model, the agents not just have different ad hoc specified opinions, but face information imperfections directly following from their place in the trading process.

The impact of unobserved fundamentals on the exchange rate determination has been reflected in a number of papers that introduce learning into originally full information models. Lewis (1989) studied an extension of the standard monetary exchange rate model and assumed a rational learning process for the true currency return drift. This approach allowed her to derive a reduced form equation for the exchange rate fundamental that was able to explain more of the U.S. dollar volatility of the 1980s than the benchmark model. This learning-from-errors model uses an ad hoc specification of fundamentals and information structure, and does not say anything on either risk pricing or the market structure origins of the estimation errors. A model of exchange rate misperception and noise-filtering with a deeper microfoundation, can be found in Gourinchas and Tornell (2002). The model works with a domestic and a foreign pricing kernel whose parameters are estimated by the agents. Estimation errors lead to under-reaction to the interest rate innovations. The authors abstract from the usual risk premium, so that the UIP violation is explained by the pricing kernel forecast errors alone. Bayesian learning extends the standard rational expectations set-up, and the price formation mechanism as such is not addressed. The result on the limit value of the forward premium crucially depends on the AR(1)-specification of the shock process.

1.1.4. Exchange rate volatility under dispersed information

The well-known failure of standard international macroeconomics to account for the observed high volatility of nominal exchange rates has lead to studies that seek to explain this volatility by asymmetric information effects. Jeanne and Rose (2002) offer an extension of the classic monetary model in which multiple equilibria are possible due to noise traders (the agents who do not fully observe the expected excess return on foreign bonds). The exogenous source of uncertainty in the model is an ad hoc shock to the purchasing power parity. The market with non-fundamental traders is modeled along the lines of DeLong et al. (1990). Jeanne and Rose (2002) treat the asymmetric information and multiple equilibria as non-fundamental phenomena that can be overcome by the right policy. This is different from our model, where the agents face both multiple fundamental uncertainties, which they have to resolve gradually by learning, and multiple potential realizations of the aggregate trading process, which is an additional risk factor and an additional source of learning (from trades) at the same time.

Another recent exchange rate model with a prominent role for noise trading is Bacchetta and van Wincoop (2003). These authors find that dispersed information magnifies non-fundamental trade component of the exchange rate volatility and leads to rational confusion—between fundamentals and noise—about its source. Since their model of the forex is Walrasian, with no explicit market structure or trading mechanism, an “infinite regress” construction of individual price expectations is employed to account for heterogeneous information. In my model, the price-setters (market-making dealers) condition on what they observe in real time by quoting prices and receiving orders. Therefore, they do not need to condition on the market-clearing price (and on the others’ expectations of the market-clearing price, and so on, generating the said infinite regress problem). They also do not need to know the exact parameters of the price process, which Bacchetta and van Wincoop have to assume to obtain a solution.

Both named papers belong to the noisy rational expectations vein. The agents are not assigned distinct roles in the trading process and their information sets and actions are not defined in terms of quotes and orders. Moreover, the outcome of these two models crucially depends on the specified behavior of noise traders. On the contrary, the results of our model are determined by fully rational behavior of the market participants who do their best under information imperfections inherent in the decentralized dealer market.

The paper is structured as follows. Section 2 describes the model variables, the structure of uncertainty and the dealer's optimization problem (Section 2.1), after which the optimal dealer strategies for passive (at own quotes) and active (at other dealers' quotes) FX trades are derived (Section 2.2). I also show how a non-zero bid–ask spread and a competitive mid-quote arise as a result of dealer optimization. Section 3 analyzes the long-run FX transaction price dynamic resulting from the UTRP. Section 4 discusses order-flow interpretation and updating of beliefs by dealers, leading to deviations from the uncovered parity of returns. Section 5 concludes. Appendix A contains proofs of technical statements, Appendix B presents some FX data and simulation experiments that support the conclusions of the model.

2. The investor–dealer decision problem

2.1. The economy

The world is split into the home country with legal tender M and the foreign country with legal tender I. By the exchange rate, we understand the price of I in M-terms. Investors are price takers in all markets. Residents of the home country account their operations in M, those residing in the foreign country use the accounting unit I.[4] Among the investors (of both residencies), there is a subset of dealers who offer FX market maker services to the others. They provide an ask quote p^a which is the price at which they agree to sell one unit, and a bid price p^b at which they agree to buy one unit, of I in exchange for M. They do not take into account the impact of own quotes on the resulting market price. Each dealer has a local customer base of those investors who only exercise FX trades through him at given quotes (they are free to vary the traded quantity from zero to infinity depending on the quotes they see). Alongside them, there are other (global) investors who are able to search for the best quote among all dealers, at a cost. Each dealer also has the ability of costly discovery of the best quote among other dealers. One would expect the investor–dealers to face a lower cost of quote search than investors without the dealing capacity. On the other hand, the non-dealer investors may know more about a certain fundamental than the dealer (information imperfections of the dealers are discussed in Section 4). This balance between costs and benefits of market making should provide an intuitive justification for the existence of dealers in the model. In the case of a small FX market (e.g. a European currency not belonging to the euro zone or an emerging economy currency), one can loosely associate global investors with multinationals and international financial corporations present in the local economy.

The order flow that a dealer observes consists of three parts. The first is the supply of I or demand for I from his local customers, dependent only on his own quote values p^a and p^b and the economic fundamentals influencing the customer base. The

[4] The present model will accommodate the Jensen inequality-related properties of the expected exchange rate returns (will be free of the residency-based asymmetry of the latter, arising in naïve no-arbitrage settings and known as Siegel's paradox). See Derviz (2002) for the details of Siegel's paradox resolution in the corresponding class of models.

remaining two order flow components come from the inter-dealer market. These are the orders of those (a) global investors and (b) other dealers, whose quote search has resulted in the choice of this dealer for the desired trade in the given period. For every party involved, participation in the inter-dealer market is a source of an information update about the statistics of the fundamental variables and the market-wide order flow, read off the received trades. The inter-dealer market is only partially transparent in the sense that each participant only observes his/her own trades, whereas the knowledge of standing best quotes is the result of a costly search for each outgoing trade. These search costs are a part of the overall transaction costs (to be defined below). One can think of an agent with access to electronic and voice brokers who service parts of the market, but who is not sure whether a better price could be found by approaching certain dealers individually. While not modeling these tradeoffs between a broker and individual search explicitly, we adopt the understanding that the best price can always be found, even if the resources dedicated to the search grow faster than the desired transaction amount.

The international investor exercising the function of an FX-dealer will be modeled as an agent characterized by four state variables. These are: x^0—domestic cash holdings, x^{d}—holdings of the domestic stock, x^i—foreign cash holdings, x^{f}—holdings of the foreign stock. The domestic stock pays out a random rate of return $\mathrm{d}r^{\mathrm{d}}$ in M-units. I also assume that maintaining the investment in x^{d} requires continuous inputs of funds (management costs) without which the value x^{d} deteriorates at a stochastic rate $\mathrm{d}\alpha^{\mathrm{d}}$. (This is done for the sake of convenience, allowing one to consider equilibria with constant average levels of asset holdings.) Let one unit of x^{d} cost X^{d} units of M. Analogous values for the foreign stock are denoted by $\mathrm{d}r^{\mathrm{f}}$, $\mathrm{d}\alpha^{\mathrm{f}}$ and X^{f}, respectively (the latter price is in $\mathtt{I}$-units). A generalization to multiple domestic and foreign assets is straightforward.

Cash holdings x^0 and x^i earn their own rates of return $\mathrm{d}r^0$ and $\mathrm{d}r^i$, respectively. In the simplest variant, it can be the overnight money market rate (or its proportional part if the elementary period is intra-day). More realistically, one should include in the cash variables the inter-bank loan/deposit positions and FX swap positions. In that case, $\mathrm{d}r^0\,(\mathrm{d}r^i)$ is the instantaneous rate paid in the domestic (foreign) money market on the corresponding portion of $x^0\,(x^i)$, i.e. the generator short rate of the domestic (foreign) term structure.

For any strictly positive Itô process z, $\mathrm{d}z/z$ will be a shortened to $\hat{\mathrm{d}}z$. The drift and diffusion coefficients of the asset return and price processes introduced above will be denoted as follows:

$$\mathrm{d}r^0 = i^{\mathrm{d}}\,\mathrm{d}t + I^{\mathrm{d}}\,\mathrm{d}Z, \quad \mathrm{d}r^i = i^{\mathrm{f}}\,\mathrm{d}t + I^{\mathrm{f}}\,\mathrm{d}Z, \quad \mathrm{d}r^{\mathrm{d}} = n^{\mathrm{d}}\,\mathrm{d}t + N^{\mathrm{d}}\,\mathrm{d}Z,$$

$$\mathrm{d}r^{\mathrm{f}} = n^{\mathrm{f}}\,\mathrm{d}t + N^{\mathrm{f}}\,\mathrm{d}Z, \quad \mathrm{d}\alpha^{\mathrm{d}} = a^{\mathrm{d}}\,\mathrm{d}t + A^{\mathrm{d}}\,\mathrm{d}Z, \quad \mathrm{d}\alpha^{\mathrm{f}} = a^{\mathrm{f}}\,\mathrm{d}t + A^{\mathrm{f}}\,\mathrm{d}Z,$$

$$\mathrm{d}X^{\mathrm{d}} = \pi^{\mathrm{d}}\,\mathrm{d}t + \rho^{\mathrm{d}}\,\mathrm{d}Z, \quad \mathrm{d}X^{\mathrm{f}} = \pi^{\mathrm{f}}\,\mathrm{d}t + \rho^{\mathrm{f}}\,\mathrm{d}Z.$$

Since it is not the purpose of this paper to study GARCH or other non-constant variance effects in asset prices, all diffusion coefficients I, N, A and ρ are assumed constant.

Itô processes r^0, r^{d}, r^i, r^{f}, α^{d} and α^{f} belong to the exogenous sources of uncertainty in the model. These processes, together with two more to be introduced later in

Sections 2 and 4, are assumed to be locally L_2-integrable on $[0, \infty)$ with respect to the physical probability $\mathbf{P}$, and generate the information filter $\mathbf{F} = (\mathsf{F}_t)_{t \geqslant 0}$. The latter must be augmented by all $\mathbf{P}$-null sets. All considered stochastic processes will be adapted to $\mathbf{F}$ (for every t, F_t denotes the partition of the event space corresponding to the full information about the operation of the markets; the probability measure $\mathbf{P}$ is defined on F_t for all t.) Let the diffusions adapted to $\mathbf{F}$ be spanned by a vector Z of mutually independent standard Brownian motions.

2.1.1. *Current FX transaction price*

We have assumed that the global market users (dealers or not) are allowed to search in the inter-dealer market for acceptable quotes. Next, we introduce the notion of the best attainable bid and ask prices for a given market user, P^{b} and P^{a}, good for one unit of $\mathtt{I}$. Trade size-dependent non-linearities will be accommodated by means of convex transaction costs (see below).

Assumption 1. For each global investor and each dealer, the best individually attainable (highest) bid price is equal to the best attainable (lowest) ask price: $P^{\mathrm{b}} = P^{\mathrm{a}} = P$, i.e. the inside spread/touch is zero.

Although, in reality, the touch is positive, it is very small under normal circumstances (i.e. unless markets become turbulent). Goodhart et al. (1996), observe that inter-dealer FX spreads on major currency pairs are usually as tiny as 1 pip or $\frac{1}{10}$ of the basis point for standard amounts. These are the numbers reported by the Reuters D2000-2 electronic brokerage system, i.e. the inside spreads, giving a perfect match with Assumption 1 (which just sets the minimal "technical" touch amount to zero for simplicity).

On the theoretical side, the inside spread diminishing with the number of competing dealers is one of the outcomes of the model by Ho and Stoll (1983). Their model also predicts that spreads of individual dealers would not fall to zero even when their number becomes large. This is in line with the results to be obtained in the present paper. More generally, the intuitive justification of Assumption 1 is related to the competitiveness of the modeled inter-dealer market. If one observed $P^{\mathrm{b}} > P^{\mathrm{a}}$, then there would be a clear arbitrage opportunity. In the opposite case $P^{\mathrm{b}} < P^{\mathrm{a}}$, there would be no inter-dealer trades at all until the too high asks went down and/or the too low bids went up to meet each other and provide gains from trade to those who seize the opportunity. There shall always be an agent who discovers the deadlocked market and undercuts/overbids the standing quote in his favor. As soon as all the arbitrage and market share appropriation possibilities have been exhausted, there remains a group of inter-dealer market sellers and another group of buyers. They transact at the mutually acceptable price P. The buyers' ask price and the sellers' bid price quotes are non-competitive: above P in the first group and below P in the second. Therefore, the buyers do not have to sell and the sellers do not have to buy.[5] The last-observed inter-dealer transaction price P

[5] Thus, the present model incorporates the observation by Silber (1984) that dealers are typically competitive on only one side of the market.

will be identified with the market price. It is the one used in the dealer's accounting. It is as a strictly positive Itô process with the law of motion $\widehat{\mathrm{d}}P = \pi\,\mathrm{d}t + \rho\,\mathrm{d}Z$, with coefficients π and ρ to be determined in equilibrium.

The dealer population is split into sellers and buyers at every given moment. One reason why someone becomes a seller and someone else a buyer is the heterogeneity of asset endowments. Even in the absence of the latter, there is a possibility of differences in privately accounted net returns on assets between dealers. For instance, foreign residents may have a handicap with regard to domestic asset management, including noisy information on returns, taxation or profit repatriation costs, and vice versa. Domestic dealers may enjoy privileged access to the domestic money market, with a resulting higher effective return on domestic cash balances, and a symmetric advantage may be enjoyed by foreign residents with respect to foreign cash. In addition, there may be information asymmetries (cf. Brennan and Cao, 1997, or Seasholes, 2000).

2.1.2. Investor performance measure

For the purpose of defining preference over the paths of investor–dealer's actions, we are introducing the dividend rate δ to be accounted for in every infinitesimal period $\mathrm{d}t$ (it is an analogue of consumption rate in standard portfolio models). This rate shall be understood as an imaginary infinitesimal contribution of the dealer department to the dividend fund of the firm, whose integral, in reality, is being withdrawn from the cash balance at discrete intervals. This infinitesimal dividend rate is a performance measure that makes the dealer accountable for the forex trade activity of the firm. In this way, the agent's preferences are made formally similar to the standard portfolio optimization setups.

2.1.3. Price search and order execution costs

In order to generate non-trivial supply and demand schedules (for instance, Glosten (1989) uses non-linear pricing schedules in his investor–dealer game in both monopolistic and competitive dealer settings), it is usual to define transaction costs that grow in a convex way in the transaction volume. It is well known that observable transaction costs incurred by a big real-life investor, let alone a dealer, are usually negligible for small/standard amounts, but grow rapidly when the trade size exceeds the standard. Accordingly, non-linear pricing schedules are a reality. In addition, some form of generalized transaction costs seems to lie behind the dealer behavior, even if these costs are not readily measured. In this paper, we take the view that these costs include the cost of a quote search in the inter-dealer market. Specifically, let us assume that in order to buy/sell volume $v^i\mathrm{d}t$ of $\mathtt{I}$ at market price P (which is the last observed highest bid if $v^i<0$ and the last observed lowest ask if $v^i>0$), one pays/receives $Pk(v^i)\mathrm{d}t$ units of M. Here, k is a strictly increasing and strictly convex function with $k'(0)=1$. That is, both buyers and sellers always pay a margin over $P|v^i|\mathrm{d}t$ in terms of transaction costs, and this margin is an increasing function of volume v^i. Two natural examples would be a quadratic function of the form $k(v)=v+k_0v^2/2$ or an exponential function $k(v)=(1/k_0)(\mathrm{e}^{k_0v}-1)$. In both cases, the positive constant k_0 expresses the degree of convexity of function k. In other words, with bigger k_0, the transaction costs grow faster with the order volume.

The defined price search cost mechanism is not specific to the forex and should be expected to exist in other decentralized markets as well. Although we do not model the market structure for the two national stocks, it is assumed that analogous transaction functions are defined for these two market segments. That is, if amounts v^{d} and v^{f} of the domestic and foreign stock are purchased per period (sold if negative), the M- and I-balances are reduced by amount $X^{\mathrm{d}}k(v^{\mathrm{d}})\,\mathrm{d}t$ and $X^{\mathrm{f}}k(v^{\mathrm{f}})\,\mathrm{d}t$, respectively.

2.1.4. The dealer's order flow

As was mentioned in the introduction, it is difficult to obtain separate data on the customer and inter-dealer components of a dealer's order book. Instead, we distinguish between the local customers of the given dealer (do not have access to or exercise FX trades with others) and the participants of the inter-dealer market. Both the global investors who search among available dealer quotes and dealers themselves belong there. Accordingly, the pair of ask and bid I-prices p^{a}, p^{b}, announced by the dealer, shall service everyone at the moment of announcement. However, we assume that the dealer is able to distinguish between a locally riskless (non-diffusion) demand $D(p^{\mathrm{a}})\,\mathrm{d}t$, a locally riskless supply $S(p^{\mathrm{b}})\,\mathrm{d}t$, and a random flow, which can go both ways, $\mathrm{d}\Lambda/P=(l\,\mathrm{d}t+\lambda\,\mathrm{d}Z)/P$, generated by Itô process Λ with drift l and diffusion λ. Altogether, the accounted change in the dealer's M-balance, induced by trades at his quotes, is equal to

$$(p^{\mathrm{a}}D(p^{\mathrm{a}})-p^{\mathrm{b}}S(p^{\mathrm{b}}))\,\mathrm{d}t+l\,\mathrm{d}t+\lambda\,\mathrm{d}Z=(p^{\mathrm{a}}D(p^{\mathrm{a}})-p^{\mathrm{b}}S(p^{\mathrm{b}}))\,\mathrm{d}t+\mathrm{d}\Lambda, \tag{1}$$

while the corresponding change in the I-balance is

$$[S(p^{\mathrm{b}})-D(p^{\mathrm{a}})]\,\mathrm{d}t-\frac{l\,\mathrm{d}t+\lambda\,\mathrm{d}Z}{P}=[S(p^{\mathrm{b}})-D(p^{\mathrm{a}})]\,\mathrm{d}t-\frac{\mathrm{d}\Lambda}{P}. \tag{2}$$

Observe, in particular, the last diffusion terms in (1) and (2) coming from the diffusion part of Λ: $\lambda\,\mathrm{d}Z$ and $-\lambda\,\mathrm{d}Z/P$. They reflect the dealer's uncertainty about the order flow from both global investors and other dealers. Indeed, one can generate the dealer's diffusion-type order flow as a limit of two small-interval discrete flows of very small purchases at p^{a} and very small sales at p^{b} with a non-zero degree of randomness in the direction of the trade and its exact magnitude. Then the drift parts of the limit flows will correspond to the demand and supply that are certain during the infinitesimal interval $\mathrm{d}t$, namely $[D(p^{\mathrm{a}})+l^{\mathrm{a}}/P]\,\mathrm{d}t$ and $[S(p^{\mathrm{b}})-l^{\mathrm{b}}/P]\,\mathrm{d}t$, where $l^{\mathrm{a}}+l^{\mathrm{b}}=l$. The diffusion part reflects the remaining uncertainty. It can be shown that the I-position in that case evolves exactly as is shown by the last term in Eq. (2).[6] The presence of the term $-l/P$ in the drift component of the order flow is explained by the possibility of the probability revision by the dealer (the drift change in Λ in accordance with the Girsanov theorem, see Elliott, 1982). The consequences of such a probability measure change in the case of a Bayesian update of beliefs will be discussed in Section 4.

[6] The said transition to the limit of discrete processes is a generalization of the well-known procedure of generating a geometric Brownian motion as a limit of random walks with a drift, when the step size goes to zero, see e.g. Dixit and Pindyck (1994, Section 3.2).

Summing up the definitions given above, we come to the following system of state-transition equations for the state vector $x = [x^0, x^{\mathrm{d}}, x^i, x^{\mathrm{f}}]^{\mathsf{T}}$ of an international investor with the dealer function (with domestic residence):

$$\begin{aligned} \mathrm{d}x^0 &= x^0\,\mathrm{d}r^0 + x^{\mathrm{d}}\,\mathrm{d}r^{\mathrm{d}} - \delta\,\mathrm{d}t - X^{\mathrm{d}}k(v^{\mathrm{d}})\,\mathrm{d}t \\ &\qquad -Pk(v^i)\,\mathrm{d}t + (p^{\mathrm{a}}D(p^{\mathrm{a}}) - p^{\mathrm{b}}S(p^{\mathrm{b}}))\,\mathrm{d}t + l\,\mathrm{d}t + \lambda\,\mathrm{d}Z, \end{aligned} \tag{3a}$$

$$\mathrm{d}x^{\mathrm{d}} = -x^{\mathrm{d}}\,\mathrm{d}\alpha^{\mathrm{d}} + v^{\mathrm{d}}\,\mathrm{d}t, \tag{3b}$$

$$\mathrm{d}x^i = x^i\,\mathrm{d}r^i + x^{\mathrm{f}}\,\mathrm{d}r^{\mathrm{f}} - X^{\mathrm{f}}\,k(v^{\mathrm{f}})\,\mathrm{d}t + v^i\,\mathrm{d}t + [S(p^{\mathrm{b}}) - D(p^{\mathrm{a}})]\,\mathrm{d}t - \frac{l\,\mathrm{d}t + \lambda\,\mathrm{d}Z}{P}, \tag{3c}$$

$$\mathrm{d}x^{\mathrm{f}} = -x^{\mathrm{f}}\,\mathrm{d}\alpha^{\mathrm{f}} + v^{\mathrm{f}}\,\mathrm{d}t. \tag{3d}$$

2.1.5. Preferences

Let there exist a function u satisfying the conventional growth and concavity conditions, such that $u(\delta)$ measures the period utility (of the shareholders), derived from receiving contribution δ to the dividend fund. The model is intended to be applicable to sufficiently long effective horizons (longer than one day), so we introduce an instantaneous time preference rate. I understand this rate, denoted by β, as the parameter of a Poisson "death" process. The present model neither requires a dealer to close the FX position in a predetermined finite time, nor does it rely on the existence of an exogenous final/underlying value of the currency. Instead, the following construction will replace the artificial shortcut of the "true liquidation value under discovery", often utilized in microstructure models.

For each time moment t, the dealer will have to close down his forex trade business within the next time interval $\mathrm{d}t$, with probability $1 - \mathrm{e}^{\beta\,\mathrm{d}t}$, and liquidate the outstanding engagement in currency I at current prices. The result of liquidation, i.e. the balance $x^0 + Px^i$, is evaluated by means of a strictly increasing and concave *exit* (*or bequest*) *utility* function G. As a result, at any time t the domestically resident dealer maximizes

$$E\left[\int_t^{\infty} \mathrm{e}^{-\beta(s-t)}\,\left\{u(\delta_s) + \beta G(x_s^0 + P_s x_s^i)\right\}\,\mathrm{d}s \middle| F_t\right] \tag{4}$$

with respect to control path $s \mapsto (\delta_s, v_s^{\mathrm{d}}, v_s^i, v_s^{\mathrm{f}}, p_s^{\mathrm{a}}, p_s^{\mathrm{b}})$, $s \geqslant t$, subject to (3), given the current values x_t of asset holdings. Note the appearance of parameter β in front of G in the integrand in (4).[7] Since, in each period $\mathrm{d}s$, the firm liquidates with probability $1 - \mathrm{e}^{-\beta\,\mathrm{d}t}$, the expected utility derived from the liquidated position is equal to

$$(1 - \mathrm{e}^{-\beta\,\mathrm{d}s})\,G_s = \frac{1 - \mathrm{e}^{-\beta\,\mathrm{d}s}}{\mathrm{d}s}\,G_s\,\mathrm{d}s \approx \beta G_s\,\mathrm{d}s.$$

[7] A similar construction of position liquidation with a positive probability, giving rise to a natural discount factor, was utilized by Foucault (1999) in a discrete time brokered trading model.

If the dealer resides abroad, her liquidation balance entering G shall be accounted for in I-units and be defined as $x^0/P + x^i$. In either case, we shall assume that G decreases to $-\infty$ when cash balances become increasingly negative. Then, the presence of G in the period utility prohibits negative cash bubbles of Ponzi type, accomplishing the same objective as the transversality condition of traditional models.

2.2. Optimal policies of the dealer

Following the results of Appendix A (see also the literature references therein), we can characterize the solution to problem (3), (4) by means of the shadow prices ξ of the four held assets, which are the adjoint processes of the problem appearing in the maximum principle. Given the Hamiltonian of the problem, as calculated in Appendix A, the optimal actions of the dealer are characterized by the following first-order conditions:

$$\delta = g(\xi_0), \tag{5a}$$

$$k'(v^i) = \frac{\xi_i}{\xi_0 P}, \tag{5b}$$

$$k'(v^{\mathrm{d}}) = \frac{\xi_{\mathrm{d}}}{\xi_0 X^{\mathrm{d}}} \tag{5c}$$

$$k'(v^{\mathrm{f}}) = \frac{\xi_{\mathrm{f}}}{\xi_i X^{\mathrm{f}}}, \tag{5d}$$

$$p^{\mathrm{a}}\left(1 - \frac{1}{\varepsilon^{\mathrm{a}}}\right) = \frac{\xi_i}{\xi_0}, \qquad p^{\mathrm{b}}\left(1 + \frac{1}{\varepsilon^{\mathrm{b}}}\right) = \frac{\xi_i}{\xi_0}, \tag{5e}$$

where g denotes the inverse function to u', $\varepsilon^{\mathrm{a}} = -p^{\mathrm{a}}D'(p^{\mathrm{a}})/D(p^{\mathrm{a}})$ is the price elasticity of the local I-demand and $\varepsilon^{\mathrm{b}} = p^{\mathrm{b}}S'(p^{\mathrm{b}})/S(p^{\mathrm{b}})$ is the price elasticity of the local I-supply, both observed by the dealer.

Condition (5a) corresponds to the standard consumption/investment optimization result $u'(\delta) = \xi_0$—the equality of the marginal utility of consumption and the marginal indirect utility of wealth. The role of consumption is played by the dividend and ξ_0 is the marginal utility of the home cash wealth. Conditions (5b)–(5d) can be understood as inverse demand schedules. For instance, (5b) can be written as $P = (\xi_i/\xi_0)(1/k'(v^i))$. The foreign-to-domestic cash shadow price ratio ξ_i/ξ_0 is the intercept of the demand function. It is the price level at which the investor initiates no transactions. Market prices higher than the shadow price ratio induce sales ($v^i < 0$) and those lower than this ratio—purchases ($v^i > 0$) of foreign currency.

According to the adjoint (Euler) equations stated in the maximum principle of Appendix A, the shadow asset prices faced by the dealer satisfy the system of equations

$$\mathrm{d}\xi_0 = \xi_0(\beta\,\mathrm{d}t - \mathrm{d}r^0 - l\,\mathrm{d}t + |I^{\mathrm{d}}|^2\,\mathrm{d}t) - \beta G\,\mathrm{d}t, \tag{6a}$$

$$\mathrm{d}\xi_{\mathrm{d}} = \xi_0(-\mathrm{d}r^{\mathrm{d}} + (I^{\mathrm{d}} - A^{\mathrm{d}})/N^{\mathrm{d}}\,\mathrm{d}t) + \xi_{\mathrm{d}}(\beta\,\mathrm{d}t + \mathrm{d}\alpha^{\mathrm{d}} + |A^{\mathrm{d}}|^2\,\mathrm{d}t), \tag{6b}$$

$$\mathrm{d}\xi_i = \xi_i(\beta\,\mathrm{d}t - \mathrm{d}r^i + |I^{\mathrm{f}}|^2\,\mathrm{d}t) - \beta PG'\,\mathrm{d}t, \tag{6c}$$

$$\mathrm{d}\xi_{\mathrm{f}} = \xi_i(-\mathrm{d}r^{\mathrm{f}} + (I^{\mathrm{f}} - A^{\mathrm{f}})N^{\mathrm{f}}\,\mathrm{d}t) + \xi_{\mathrm{f}}(\beta\,\mathrm{d}t + \mathrm{d}\alpha^{\mathrm{f}} + |A^{\mathrm{f}}|^2\,\mathrm{d}t). \tag{6d}$$

If a dealer resides in the foreign country, she faces the shadow prices whose laws of motion must be a symmetrically adjusted version of (6), in accordance with her units of account. In other words, in Eqs. (6a)–(6d), one would have to switch the index pairs (0, d) and (i, f). Only Eq. (5e), characterizing the quote setting, would remain as it is.

Beside the mirror-image Euler equations, we allow for other resident vs. non-resident dealer differences. As already mentioned in 2.1, stock ownership by a non-resident may involve a different set of dr- and dα-components of the stock return compared to a resident (e.g. higher costs of monitoring, tax benefits, legal expenses, etc.). If the dealers lack precise knowledge of the distribution of uncertainty factors (the situation to be studied in Section 4), there may be also differences in the information sets. Specifically, the expected return on foreign stock may be estimated with a model that differs from the one used by the local resident. The result is, again, a wedge in the parameter values of (6) between resident and non-resident dealers.

Whatever the source of dealer heterogeneity, its result is a non-zero FX order flow. In this paper, we do not look for an explicit map between legal status, endowment, information or other heterogeneities, and the equilibrium order flow. It is unlikely that one can exhaust all the important dealer asymmetry cases by constructing explicitly solvable models. Instead, we shall later look for the results stated in terms of the aggregate order flow, taking it as a sufficient statistic for the possible heterogeneities.

2.2.1. *Spread*

An immediate consequence of the f.o.c. (5e) is the following expression for the dealer's bid–ask spread:

$$\frac{p^{\mathrm{a}}}{p^{\mathrm{b}}} = \frac{1 + 1/\varepsilon^{\mathrm{a}}}{1 - 1/\varepsilon^{\mathrm{b}}}. \tag{7}$$

If elasticities ε^{a} and ε^{b} are high, the log of the right-hand side of (7) is approximately equal to $1/\varepsilon^{\mathrm{a}} + 1/\varepsilon^{\mathrm{b}}$. The conjecture about the high-price elasticities of demand and supply on the side of the dealer's customer base seems justified, since the customers are not facing a unique dealer-monopolist. It is reasonable to assume that as soon as the disadvantage of trading with a monopolist or monopsonist becomes too evident, a customer can always try to look up another one, i.e. turn "global" despite the associated costs. In the sequel, I will assume the existence of a common upper bound for inverse elasticities $1/\varepsilon^{\mathrm{a}}$ and $1/\varepsilon^{\mathrm{b}}$ for all customer bases.

Eq. (7) would be a standard outcome in any monopolist two-way market maker problem, regardless of the presence of uncertainty. The two less standard elements of the present model are: time-variable and stochastic spreads, and the equilibrium paths of dealer quotes that reflect information about fundamentals, to the extent the latter is disseminated by other parties' actions (see Section 4 below). The latter feature is achieved by considering a competitive inter-dealer market.

To generate realistic variable FX spreads, it is necessary to consider only supply and demand schedules with non-constant non-deterministic price elasticities. Typical functional forms for functions S and D, indeed, satisfy this requirement. To see this, observe that any investor can be modeled by means of equations analogous to (3) and (4) above, if one eliminates the dealer order flow parts from transition Eqs. (3a) and (3c). Then the FX supply/demand schedule will be given by an equation formally identical to (5b). Depending on the current values of shadow prices relative to the offered market I-price, some investors will be sellers and others buyers. In both cases, the aggregate S and D that the dealer sees will be linear functions of the inverse price with stochastic coefficients. Such functions give rise to variable stochastic price elasticities.

The laws of motion (6) of the shadow asset prices characterize them in terms of the *fundamental variables* of the defined economy. These fundamentals in the present model are comprised of various components of the total asset returns (dividends, price movements, depreciation rates), plus a variable which characterizes the aggregate flow of I-funds to/from the domestic dealers (to be defined in Section 4). The fundamental information accessible to different groups of market participants can have different quality. Possibilities to model information dissemination processes with regard to fundamentals within the present approach will be discussed in Section 4. But first, to establish that dealer quotes reflect the processed information on fundamentals competitively (therewith reducing the monopolistic welfare loss effects), it is necessary to deduce a number of properties of possible market equilibria. This is done in the next section.

3. Shadow price parity, uncovered parity of total returns, and the asymptotic dynamic of the exchange rate

The first-order conditions of optimality (5) can be used to state a fairly general property of the exchange rate dynamics, which can be called the generalized UTRP condition. The underlying property of equilibrium in the asset markets is the same as the one that leads to the consumption-based CAPM. A similar result would be obtained in most international portfolio optimization models. In contrast to the very much discredited uncovered interest rate parity of the textbooks, the total return parity works with rates of return on financial instruments other than the short-term money market rates. The resulting formulae contain expectations of domestic and foreign returns that are related to the expected change of the foreign currency value, all three values being uncertain. That is, instead of testing the predictive ability of the forward rate, which is known at the start of the period, these other approaches investigate correlation between two stochastic processes: the cross-currency return difference and the FX return. The instruments employed in the analysis are usually long-term government bonds (Nadal-De Simone and Razzak, 1999; Berk and Knot, 2001). Sometimes, the actual time horizon is replaced by duration (Alexius, 2001). Derviz (2002) finds that there are periods when the long government bond yield-based parity is verified for a small open economy currency, with structural breaks caused by changes in market participant composition. (Appendix B discusses how the present model is able to complement the analysis of that paper.) There is also a wide understanding among practitioners within the monetary authorities that the interest rates one needs to look at to gauge market exchange rate expectations are, actually, rates of return on benchmark bonds across currencies.

To formulate the result, it is necessary to define the total instantaneous rates of return on the domestic and foreign stock. In this model, the return rate is the sum of the instantaneous coupon/dividend rate relative to the current price, $\mathrm{d}r/X$, the capital gain $\mathrm{d}X/X$, less the depreciation rate $\mathrm{d}\alpha$:

$$\mathrm{d}R^{\mathrm{d}} = \frac{\mathrm{d}r^{\mathrm{d}}}{X^{\mathrm{d}}} + \widehat{\mathrm{d}}\,X^{\mathrm{d}} - \mathrm{d}\alpha^{\mathrm{d}} = y^{\mathrm{d}}\,\mathrm{d}t + \upsilon^{\mathrm{d}}\,\mathrm{d}Z,$$

$$\mathrm{d}R^{\mathrm{f}} = \frac{\mathrm{d}r^{\mathrm{f}}}{X^{\mathrm{f}}} + \widehat{\mathrm{d}}\,X^{\mathrm{f}} - \mathrm{d}\alpha^{\mathrm{f}} = y^{\mathrm{f}}\,\mathrm{d}t + \upsilon^{\mathrm{f}}\,\mathrm{d}Z.$$

Next, define the auxiliary *shadow price parity* index variable Y as

$$Y = \frac{\xi_{\mathrm{f}} X^{\mathrm{d}}}{\xi_{\mathrm{d}} P X^{\mathrm{f}}}.$$

The reason for the name is the fact that, according to Itô's lemma applied to Eqs. (6b) and (6d) for the shadow prices, its stochastic differential satisfies the following property:

$$\widehat{\mathrm{d}}\,Y = \mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}} - \widehat{\mathrm{d}}\,P + A\,\mathrm{d}t. \tag{8}$$

Here, A (the *disparity* term) is the sum of a number of covariance terms (they come about as a consequence of Itô's lemma). They are usually small and, according to our assumption on the constancy of diffusion coefficients of exogenous variables, A is a constant.

The first three terms on the right-hand side of (8) define the *uncovered parity of total returns* (UTRP) for the exchange rate P. Namely, if Y were a constant and the disparity term A close to zero, then the relative expected change in P between times t and $t+\mathrm{d}t$ would be equal to the total return differential. Thus, the uncovered return parity can be formulated as the equality of the exchange rate return to the instantaneous return rate differential between the domestic and foreign stock plus a covariance term

$$\widehat{\mathrm{d}}\,P = \mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}} + A\,\mathrm{d}t. \tag{9}$$

If we had multiple assets denominated in each currency in the model, then an analogue of (9) would be valid for any cross-currency pair.

Using the investor first-order optimality condition (5b), we can now take a first intuitive look at why the FX order flow emerges simultaneously with the deviations from uncovered parity. Denote the shadow cash price ratio ξ_i/ξ_0 of the given market participant by S. This is a "no-trade" level of the exchange rate and also the "shadow exchange rate" that satisfies the uncovered parity condition exactly, as can be derived from Euler equations (6). Any deviation of the exchange rate from uncovered parity (i.e. of P from S) causes a non-zero order flow ($k^i = k'(v^i) \neq 1$ for $v^i = 0$).

Note that the uncovered return parity equation (9) is symmetric with respect to the residence country of the investor. In other words, this uncovered parity, as opposed to the traditional uncovered interest rate parity, is free of the Jensen inequality effects (Siegel's paradox).[8]

By introducing the shortening notations $k^{\mathrm{d}} = k'(v^{\mathrm{d}})$, $k^i = k'(v^i)$, $k^{\mathrm{f}} = k'(v^{\mathrm{f}})$, we derive from (5b) to (5e) that under optimal investor actions Y is equal to

$$Y = \frac{k^i k^{\mathrm{f}}}{k^{\mathrm{d}}}. \tag{10}$$

If the markets were characterized by the existence of a representative agent with no dealer–customer distinction (and the clearing prices were set by a Walrasian auctioneer), then all security purchase rates would have to be set to constants in equilibrium. Possible equilibria would then include ones with constant asset price trends, constant Y and the exchange rate that satisfies the uncovered return parity (9) exactly. However, in our model, the shadow price parity index is not a constant.

Note that, although the *stochastic differential* (8) of the uncovered parity index may be formally the same for all market participants, different initial asset endowments lead to different initial shadow cash price values ξ_0, ξ_i and to different initial values of Y for different market participants. For instance, end-users may have different endowments than the dealers. In addition, heterogeneous information across dealers (e.g. residents vs. non-residents) results in stochastic integration of $\mathrm{d}Y$ along different paths between the initial and the current time, creating more heterogeneity in the current values of Y.

The dealer in our model makes the market in only one security (foreign cash). To make the forex analysis tractable, I need to isolate it from other assets and simplify the dealer's equilibrium portfolio management actions in other markets. To do this, I shall assume that

- the price processes X^{d} and X^{f} have no trend ($\pi^{\mathrm{d}} = \pi^{\mathrm{f}} = 0$),
- the asset holdings x^{d} and x^{f} of every dealer and investor possess long-run average limit levels $\bar{x}^{\mathrm{d}}$, $\bar{x}^{\mathrm{f}}$. Each agent simply maintains the long-run average level of both asset holdings in the portfolio by compensating, in the drift part, for their continuous attrition described by (3b) and (3d). Then the optimal purchase rates v^{d} and v^{f} are given by

$$v^{\mathrm{d}} = \bar{v}^{\mathrm{d}} = \bar{x}^{\mathrm{d}} a^{\mathrm{d}}, \qquad v^{\mathrm{f}} = \bar{v}^{\mathrm{f}} = \bar{x}^{\mathrm{f}} a^{\mathrm{f}}.$$

This makes the asset holding processes revert to $\bar{x}^{\mathrm{d}}$, $\bar{x}^{\mathrm{f}}$ in the mean

$$\mathrm{d}x^{\mathrm{d}} = a^{\mathrm{d}}(\bar{x}^{\mathrm{d}} - x^{\mathrm{d}})\,\mathrm{d}t + x^{\mathrm{d}} A^{\mathrm{d}}\,\mathrm{d}Z, \qquad \mathrm{d}x^{\mathrm{f}} = a^{\mathrm{f}}(\bar{x}^{\mathrm{f}} - x^{\mathrm{f}})\,\mathrm{d}t + x^{\mathrm{f}} A^{\mathrm{f}}\,\mathrm{d}Z.$$

By making a pure diffusion assumption about the stock prices, I imply that all demand fluctuations generated by the dealers in the stock markets are absorbed by unspecified outside investors. Naturally, if asset return trends are unobservable, as will be the case in Section 4, x^{d} and x^{f} are mean-reverting under the subjective probability,

[8] This is true since the shadow price parity index Y is residency-invariant: by switching the roles of superscripts d and f and replacing P by $1/P$, one sees that a foreign investor shadow price parity index is equal to $1/Y$. Among other things, the shadow price parity index is constant for foreign investors if and only if it is constant for domestic ones. Recall that parity equation (9) was obtained by an Itô-differentiating Y. Therefore, (9) holds regardless of the country of residence, provided one corrects for the additional variance term in the disparity constant A. See Derviz (2002) for details of the UTRP model.

but have additional time-dependent trend components under the objective probability. These time-dependent terms have finite long-run limits and are not important for the asymptotic results of this section.

The other three assumptions are aimed at limiting the long-run price trajectories in the forex to the class of bounded ones. That is, the attention is restricted to economies and equilibria with the following properties:

(a) the exchange rate growth rate π is bounded from both sides;
(b) each of the earlier defined categories of market participants, namely the dealers–domestic residents, dealers–foreign residents, global investors and local customer bases of each dealer, is formed by agents with identical preferences and processing cost functions;
(c) the aggregate cumulative client currency supply and demand volumes, generated by non-dealer investors in both parts of the world, have bounded drifts and are locally riskless (i.e. contain no diffusions).

The benefit of the above restrictions is the existence of common equilibrium upper and lower bounds for individual inter-dealer orders v^i and, consequently, individual parameters k^i as well. Assumption (b) together with the no-diffusion assumption about the non-dealer aggregate I-supplies/demands (the second part of assumption (c)) will be used in Section 4.

The optimal quote equations (5e), if combined with assumptions (a)–(c), lead to a result about the long-run behavior of the dealer quotes.

Proposition 1. *Under assumptions* (a)–(c) *made above*, *bid and ask quotes* p^{b}, p^{a} *are asymptotically equivalent to*

$$\left(1 \pm \frac{1}{\varepsilon^{\mathrm{b,a}}}\right)^{-1} YP. \tag{11}$$

In the above formula, *the shadow price parity index* Y *is asymptotically a constant*.

The necessity for Y to be almost constant is clear from the existence of upper and lower bounds for k^i. Asymptote (11) itself results from substituting into (5e) the following expression for the shadow cash price ratio ξ_i/ξ_0:

$$\frac{\xi_i}{\xi_0} = Y\frac{k^{\mathrm{d}}}{k^{\mathrm{f}}}P, \tag{12}$$

which follows immediately from (5b) to (5d).

The message of Proposition 1 is the existence of a strong link between the dealer behavior towards his/her clients and the constraints coming from the inter-dealer market. Although it is optimal for the dealer to maintain a positive spread between bid and ask quotes, both "half-spreads" (i.e. the distances between p^{a} and P and between P and p^{b}) are asymptotically pinned down by the dealer's shadow price parity index. (The mid-point itself is bounded in expectation, according to assumption (a).) The latter, by definition, is tied to the shadow cash price ratio. In other words, the shadow price parity index governs the "quote shading" in the model. This index is fundamental-driven (cross-border asset return differential-driven) and is, therefore, competitively determined by all dealers and other investors. Accordingly, no dealer has an absolute monopoly power over the clients.

4. Fundamental information extraction from the order flow

As is usual in client–dealer models of FX trade, dealers in the present model learn from publicly observed events and private signals contained in the order flow. In models of investment under diffusion uncertainty in continuous time, Bayesian belief updating is implied by the dealer optimization procedure itself, as follows from the Girsanov theorem applied to the probability change in the optimal control of diffusions (see Elliott, 1982, for details). In other words, agents facing imperfectly observed asset returns follow the so-called *separation principle*. They first form conditional expectations of unobserved mean asset returns and then solve the optimization problem with these estimated returns replacing the actual unobserved ones (cf. Dothan and Feldman, 1986, or Genotte, 1986). So, I will now specify the filtering procedure applied by the dealer. After that, I obtain a formula for the equilibrium instantaneous FX return by making a more specific assumption about the nature of beliefs and signals. As a result, one will be able to formulate narrower results about the impact of belief changes regarding the fundamental asset returns and the aggregate I-funds flow on the dealer–investor actions.

Any Bayesian belief update in the present model can be characterized by a change in the probability measure Pr that a dealer uses in the optimization problem (3), (4). Process Z spanning the exogenous uncertainties is standard Brownian with mutually independent components under the reference measure Pr. Under the new probability Pr^*, it becomes a Brownian motion with a drift. More precisely, there exists another vector, Z^*, of standard mutually independent Brownian motions under Pr^*, related to Z in accordance with the Girsanov theorem. Symbolically, the relation between Z and Z^* reads

$$\mathrm{d}Z = h\,\mathrm{d}t + \mathrm{d}Z^*, \tag{13}$$

where h is an $\mathbf{F}$-adapted process satisfying a number of regularity conditions. The equilibria of the model can be associated with individual trajectories of h. The latter give rise to the inter-dealer market order flow trajectories v^i into/out of $\mathtt{I}$, directed by $\mathtt{I}$-purchasing/selling dealers towards $\mathtt{I}$-selling/purchasing ones (see the dealer's decision problem in Section 2.1). Every h also determines the trajectories of the dealers' bid and ask quotes in the buying and the selling groups and, thereby, the client order flows.[9]

The probability change generator h is determined by the filtering technique that the agent uses when processing the observations. We will assume that the agents utilize the Kalman–Bucy filter. The latter is a standard tool in the optimal control with partial observations. It is also widely applied in portfolio optimization and asset pricing models

[9] One could reduce the set of possible self-fulfilling equilibiria by deriving the aggregate order flow between asymmetric groups of dealers endogenously (although the calculations are messy). An exact quantification would mean specializing the model too much from the ouset, which we choose not to do here.

with belief updating (cf. Detemple, 1986; Dothan and Feldman, 1986; Genotte, 1986; Brennan, 1998; Zapatero, 1998, among others). Eqs. (15)–(18) list the basic properties of the dealer's Bayesian learning.

To specify the signal interpretation by the dealer, I introduce the random process M of the cumulative net outgoing purchases of foreign currency in the inter-dealer market by dealer–domestic residents. In other words, M is a measure of the cumulative inter-dealer order flow from M-residents to I-residents. Process M evolves according to

$$\mathrm{d}M = m\,\mathrm{d}t + \mu\,\mathrm{d}Z.$$

Given the homogeneity of the national dealer populations (assumption (b) of Section 3), we conclude that m is formed by the summation of identical active inter-dealer trades v^i across the M-resident dealers. An individual domestic dealer does not consider the impact of his own actions on the composition of M.

Technically, every trajectory of M corresponds to a self-fulfilling equilibrium, because one could choose a change, given by (13), of probability measure from objective to subjective, which would support it. Every jump from the reference probability to a different one corresponds to a rational confusion, if seen from the perfectly informed observer's point of view. However, most of these equilibria would be ad hoc and hard to rationalize. We shall restrict attention to such equilibria with non-zero drift m (and, accordingly, non-constant M) that would result from asymmetry among dealers, given that the global end-users split orders between them and therewith create a "fundamentally informative" component of the received order flow. (Note that if there were only local end-users for every dealer, inter-dealer orders would just mean redistribution of risks from received trades and it would be difficult to rationalize a relation of these orders with other fundamental variables in the economy.) Quantifying each possible source of asymmetry explicitly would exceed the scope of the paper. Therefore, in the sequel we take *informational asymmetry* between resident and non-resident dealers as given. This asymmetry means that dealers of different residencies observe the same fundamentals but interpret them differently. Specifically, I look at the behavior of a resident dealer who has an imperfect knowledge about the parameters of market risk processes.

Let us define the information set of the dealer. It is determined by the *observation process* $Q=[r^{\mathrm{d}},X^{\mathrm{d}},\alpha^{\mathrm{d}},r^0,r^{\mathrm{f}},X^{\mathrm{f}},\alpha^{\mathrm{f}},r^i,M]^{\mathsf{T}}$. It is a vector with components—individual sources of subjectively perceived uncertainty. Symbolically, the evolution of Q will be written as $\mathrm{d}Q = q\,\mathrm{d}t + \Theta\,\mathrm{d}Z$. This is the dynamic under the objective probability Pr, utilized by those who are able to identify the drift part $q\,\mathrm{d}t$ precisely.[10] The following assumption summarizes the usual set-up of continuous time Bayesian portfolio optimization models with partial observations (cf. Detemple, 1986; Genotte, 1986; Brennan, 1998, or Zapatero, 1998) and states it in the notation of the present model.

[10] To be able to identify Θ, it is sufficient to know only the *values* of Q across time, since, with this knowledge, the quadratic variation of Q can be computed.

Assumption 2. Every dealer correctly observes the current market fundamentals, as expressed by Q, and all dealers agree upon the values of diffusion coefficients Θ. However, dealers have imprecise information about the value of the drift term q. Consequently, an individual dealer's information filter $\mathbf{F}^Q$ generated by Q (and completed to satisfy the usual conditions) is cruder than the complete information filter $\mathbf{F}$ introduced in Section 2. Accordingly, the dealer's subjective probability measure $\Pr^*$ differs from the restriction of Pr on $\mathbf{F}^Q$.

As regards the first eight components of Q (I shall denote them by Q'), the assumption of their observability by any dealer is plausible enough and does not require extended justification. On the other hand, it might seem unnatural for a single domestic/foreign dealer to know exactly how much `I`-currency his/her compatriot population has accumulated in aggregate. Assumption 2 is, in fact, weaker. It only requires that a noisy signal comprising (a) the full list of diffusions spanning the diffusion part of M, and (b) an imprecise measure of its drift, is received at every moment. This is made possible by observations of the order flow, as explained below.

Let the components of vector Z of Brownian motions generating the risks of the economy be split into part Z^Q which spans the observations process Q, and the independent part Z^w that one needs to add in order to span the unobserved states. (The latter are responsible for the difference between $\mathbf{F}^Q$ and $\mathbf{F}$.) The dimension of Z^Q must be equal to that of Q. Moreover, the law of motion of Q shall only involve Z^Q, so that it can be written as $\mathrm{d}Q = w\,\mathrm{d}t + \Theta\,\mathrm{d}Z^Q$, with a non-singular diffusion matrix Θ satisfying regularity conditions needed for the Girsanov theorem to be applicable. Besides, I assume that the last component of Q, i.e. M, is observed through the random component Λ of the order flow received by the dealer (as defined in Section 2.1). This means, among other things, that $\mathrm{d}\Lambda$ is spanned by the totality of the components of $\mathrm{d}Z^Q$ in a non-trivial way (i.e. the private order flow is disturbed by the full range of the exchange rate-related uncertainty factors). It can be shown that $\mathrm{d}\Lambda$ gives the dealer a precise signal about the value of $\mathrm{d}M$ (although not about its statistics). Indeed, by summing up transition equations (3c) for all domestic dealers, one arrives at the aggregate relationship

$$\frac{\mathrm{d}\Lambda}{P} = \mathrm{d}M - \{\mathrm{d}x^{iT} - (x^{iT}\,\mathrm{d}r^i + x^{\mathrm{f}T}\,\mathrm{d}r^{\mathrm{f}} - X^{\mathrm{f}}K^{\mathrm{f}}\,\mathrm{d}t + c\,\mathrm{d}t)\}. \tag{14}$$

Here, $x^{\mathrm{f}T}$, x^{iT} are total foreign stock and `I`-holdings by the domestic dealer population. K^{f} is the total net purchase rate of the foreign stock by domestic dealers (it is the sum of identical k^{f} terms across the domestic dealer population). Finally, c is the net client sales rate of `I` to domestic dealers (there are no diffusions in this aggregate rate according to assumption (c) of the previous section). Invoking assumption (b) of Section 3 about the investor homogeneity inside categories, we see that the dealer knows both K^{f} (since he knows his own behavior) and c (which is equal to the sum of identical terms $S(p^{\mathrm{b}}) - D(p^{\mathrm{a}})$ across trade partners of all the domestic dealers). This means that $\mathrm{d}\Lambda$ indeed signals $\mathrm{d}M$. Eq. (14) also shows that the diffusion part of $\mathrm{d}\Lambda$ is spanned by the same vector of Brownian motions as that of $\mathrm{d}Q$. That is, the dealer transition equations (3) are consistent with the dealer's information being defined by $\mathbf{F}^Q$ (the separation principle), as stated in Assumption 2.

4.1. Bayesian learning by the dealer

Vector q of drift coefficients of the observation process Q will be viewed as an Itô process w with the law of motion

$$dw = \lfloor a^0 + Aw \rfloor\, dt + U\, dZ^Q + B\, dZ^w \tag{15}$$

under the objective probability. Here, A is a square matrix which expresses autoregression in w (note the different use of this letter compared to Sections 2 and 3). If $-A$ is positive-definite, then $-A^{-1}a^0$ is the vector of long-run values to which w reverts in the mean. B and U are diffusion coefficient matrices corresponding to the state (B) and observation (U) risk factors. All coefficients are bounded $\mathbf{F}^Q$-adapted processes, which, in addition, may be non-trivial functions of time. This is the state process unobservable directly by the dealer. Instead, each dealer has a belief about w, denoted by $\widehat{w}$, which is a conditional expectation of w given the dealer's current information: $\widehat{w}_t = E\lfloor w_t | \mathbf{F}_t^Q \rfloor$. This is an $\mathbf{F}^Q$-adapted process, with initial value $\widehat{w}_0$ and the initial variance–covariance matrix ω_0 assumed to be given. Let us denote by $\varepsilon_1 = \widehat{w}_t - w_t$ the subjective estimation error of the dealer at time t.

Put $V = \Theta \cdot \Theta^{\mathrm{T}}$. According to the properties of the Kalman–Bucy filter (see Liptser and Shiryaev, 1977, Chapter 12, for details), process $\widehat{w}$ satisfies the s.d.e.

$$\begin{aligned} d\widehat{w}_t &= (a^0 + A\widehat{w}_t)\, dt + (U\Theta^{-1} + \omega_1 V^{-1})[dQ_t - \widehat{w}_t\, dt] \\ &= \{a^0 + Aw - (U\Theta^{-1} + \omega \cdot V^{-1} - A)(\widehat{w} - w)\}\, dt \\ &\quad + \lfloor U + \omega \cdot (\Theta^{-1})^{\mathrm{T}} \rfloor\, dZ^Q \end{aligned} \tag{16}$$

under the objective probability Pr. The variance–covariance process ω satisfies the deterministic Riccati differential equation

$$\begin{aligned} \frac{d\omega}{dt} &= A \cdot \omega + \omega \cdot A^{\mathrm{T}} + B \cdot B^{\mathrm{T}} + U \cdot U^{\mathrm{T}} - [U \cdot \Theta^{\mathrm{T}} + \omega] \cdot V^{-1} \cdot [\Theta \cdot U^{\mathrm{T}} + \omega] \\ &= [A - U \cdot \Theta^{-1}]\omega + \omega\lfloor A^{\mathrm{T}} - (\Theta^{\mathrm{T}})^{-1} \cdot U^{\mathrm{T}} \rfloor + B \cdot B^{\mathrm{T}} - \omega \cdot V^{-1} \cdot \omega \end{aligned} \tag{17}$$

with initial condition ω_0.

Note that $dZ^{Q^*} = \Theta^{-1}(dQ - \widehat{w}\, dt)$ is a vector of standard mutually independent Brownian motion noises under the dealer's subjective probability Pr^* (normalization of the dealer's "innovation process"). Equivalently,

$$dZ^Q = dZ^{Q^*} + \Theta^{-1}\varepsilon\, dt \tag{18}$$

and the Girsanov measure change generator h in (13) is equal to $h = [\Theta^{-1}\varepsilon, 0]^{\mathrm{T}}$.

By subtracting (15) from (16), one obtains the s.d.e. for the dealer's estimation error

$$\begin{aligned} d\varepsilon &= (U \cdot \Theta^{-1} + \omega \cdot V^{-1} - A)\varepsilon\, dt + \lfloor U + \omega \cdot (\Theta^{-1})^{\mathrm{T}} \rfloor\, dZ^Q - B\, dZ^w \\ &= -\chi\varepsilon\, dt + [\chi + A]\Theta\, dZ^Q - B\, dZ^w. \end{aligned} \tag{19}$$

This error is a martingale under Pr and, if matrix $\chi = U\Theta^{-1} + \omega V^{-1} - A$ is stable, then with time, the subjective estimate becomes closer to the true value of w in the mean.

Recall that we assume identical dealers within each nation, and let the size of the M-resident dealer population be normalized to unity. Then, in equilibrium, the domestic dealer optimal outgoing order v^i must be equal to the drift of the aggregate domestic order flow dM going to the non-resident market-makers (the diffusion part of this order flow comes from the noise traders among the local market users, see Section 2.1). However, the dealer makes decisions under imperfect information about the parameters of process M. (Recall that, under the given information structure, the dealer solves the optimization problem from Section 2 under probability Pr* instead of Pr, and all the processes he works with are $\mathbf{F}^Q$-projections of the objective $\mathbf{F}$-adapted processes.) Therefore, he thinks that the whole domestic dealer population acts like him:

$$\widehat{m}_t = v_t^i$$

for all t. Here, $\widehat{m}$, the dealer's subjective estimate of the aggregate mean rate of active I-purchases from non-residents, is the last component of the conditional expectation $\widehat{w}$ of the unobserved state process w. Naturally, under the objective probability, the aggregate inter-dealer plus the local noise trader orders follow the law of motion

$$\widehat{m}\ \mathrm{d}t + \mu\ \mathrm{d}Z^Q = \widehat{m}\ \mathrm{d}t + \mu\ \mathrm{d}Z^{Q^*} + \mu(\mathrm{d}Z^Q - \mathrm{d}Z^{Q^*}) = \mathrm{d}M + \mu\Theta^{-1}\varepsilon\ \mathrm{d}t.$$

The second equality above follows from the measure change rule (18) and the fact that the dealer, while making a correct inference about the aggregate cross-border order flow dM from his observations, makes a mistake when splitting this flow into the drift and the diffusion part: $\mathrm{d}M = \widehat{m}\ \mathrm{d}t + \mu\ \mathrm{d}Z^{Q^*} = m\ \mathrm{d}t + \mu\ \mathrm{d}Z^Q$. His subjectively optimal decisions lead to an extra—compared to the full information case—mean order of size $\mu\Theta^{-1}\varepsilon$ per period received by non-resident dealers. This "rational confusion" with respect to optimal orders under objective probability has pricing consequences discussed below.

In the notations of Section 2, the optimal policy of the dealer implies

$$P = \frac{\xi_{\mathrm{f}} X^{\mathrm{d}}}{\xi_i X^{\mathrm{f}}} \frac{k^{\mathrm{d}}}{k^{\mathrm{f}}} \frac{1}{k^i} = \theta \frac{k^{\mathrm{d}}}{k^{\mathrm{f}}} \frac{1}{k'(v^i)} = \theta \frac{k^{\mathrm{d}}}{k^{\mathrm{f}}} \frac{1}{k'(\widehat{m})}.$$

According to the assumptions of Section 3, k^{d} and k^{f} are constants. Further, the term θ (which is related to the shadow price parity index Y) is an Itô process with differential equal to the total return differential $\mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}}$ plus a covariance term $c^0\ \mathrm{d}t$ dependent only on the components of observations covariance matrix Θ. Let $\kappa = k''(\widehat{m})/k'(\widehat{m})$. Under the linear quadratic quote search function k we get $\kappa = k_0/(1 + k_0\,\widehat{m})$ and under the exponential quote search function—simply $\kappa = k_0$ (cf. Section 2.1). Recall that $k_0 > 0$ is the transaction function parameter, which determines how fast the costs (of finding the best quote and getting one's FX trade order executed at that quote) grow with the order volume.

We are now able to formulate the principle result of the paper.

Proposition 2 (Order flow-adjusted UTRP). *The instantaneous exchange rate return in an inter-dealer FX market with imperfect dealer information about the drift terms* *of the fundamentals is equal to*

$$\mathrm{d}P = \mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}} + c^0\,\mathrm{d}t - \kappa\,\mathrm{d}\,\widehat{m} - \kappa b^m \cdot (\upsilon^{\mathrm{d}} - \upsilon^{\mathrm{f}})^{\mathrm{T}}\,\mathrm{d}t + \tfrac{1}{2}\kappa^2 |b^m|^2\,\mathrm{d}t, \tag{20}$$

$$\begin{aligned}\mathrm{d}\,\widehat{m} &= \mathrm{d}m + \mathrm{d}\varepsilon^m \\ &= \mathrm{d}m + \chi^m \varepsilon\,\mathrm{d}t + b^m\,\mathrm{d}Z^Q - B^m\,\mathrm{d}Z^w,\end{aligned} \tag{21}$$

where b^m is the last row of the diffusion matrix $b = U + \omega \cdot (\Theta^{-1})^{\mathrm{T}} = (\chi + A)\Theta$ from (16) *and B^m is the last row of **B**.*

As can be seen from the evolution law (19) for the estimation error, matrix b expresses the observation-related uncertainties driving the error process ε, whereas $-B$ is the matrix of diffusion coefficients of its state-related uncertainties. Accordingly, $|b^m|^2$ in (20) is the conditional variance of the observation-related noise in the m-estimation error ε^m, and $b^m \cdot (\upsilon^{\mathrm{d}} - \upsilon^{\mathrm{f}})^{\mathrm{T}}$ is the conditional covariance of that same observation-related noise with the return differential noise.

Eq. (20) describes the generalized UTRP in an equilibrium with imperfect dealer information and Bayesian learning from the order flow signal. Here, the first three terms on the right-hand side represent the standard UTRP: the difference of the instantaneous returns at home and abroad plus the Jensen inequality constant. Since (20) is a continuous-time equality between stochastic differentials, it must be understood as an equation for ex ante returns. Specifically, $\mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}}$ is the infinitesimal return differential over the period between t and $t+\mathrm{d}t$, unknown at time t, and $\hat{\mathrm{d}}P$ is the infinitesimal FX- (i.e. I-) return over the same period. Unlike the traditional UIP, (20)—as well as its "Walrasian" counterpart (9)—is a statement about a linear dependence of *three uncertain returns*, not the uncertain FX-return and the known interest rate differential. The drift part of this statement concerns expected returns and the diffusion part—their random innovations.

The last three terms on the right-hand side of (20) represent a deviation from UTRP, which we call *disparity* (or *country premium*). The main difference compared to standard uncovered parity formulae is the presence of the term containing the evolution of the dealer-perceived trend of the cross-border order flow, $\mathrm{d}\,\widehat{m}$. The latter can be split into $\mathrm{d}m$, the full-information law of motion of m, and the error term $\mathrm{d}\varepsilon^m = \mathrm{d}(\widehat{m} - m)$, given by the last component of (19).

Even under full symmetric information about the cross-border order flow statistics, (20) shows why the exchange rate behavior may frequently deviate from the uncovered parity rule. Any liquidity-induced movement in the aggregate order flow trend m leads to an additional component in the observed country premium. For instance, if non-residents accelerate their purchases of I, generating a positive shift of m, I-currency appreciates more than prescribed by the uncovered parity. When, in addition, the dealers are imperfectly informed about m, even the mentioned liquidity-adjusted UTRP is violated (see an example below). Formally, the term in (20) containing $\mathrm{d}(\widehat{m} - m)$ creates an additional individually perfectly rational disparity. Both effects would be impossible in a purely Walrasian FX market.

The order flow-adjusted UTRP (20) indicates that a non-stationary disparity term behavior corresponds to a period when the aggregate cross-border inter-dealer order flow $\mathrm{d}M$ has a non-zero time-dependent drift parameter m. By collecting all diffusion parameter terms in a single (close to constant) term c^1, one can rewrite (20) as

$$\widehat{\mathrm{d}}\,P = \mathrm{d}R^{\mathrm{d}} - \mathrm{d}R^{\mathrm{f}} + c^1\ \mathrm{d}t - \kappa\ \mathrm{d}\,\widehat{m}\,. \tag{22}$$

Even if one ignores the difference between objective m and the dealer's subjective $\widehat{m}$, a non-zero aggregate I-purchase or I-sale pressure resulting in a non-trivial dynamic of m is able to generate a temporary deviation from the uncovered parity rule (9). Let us assume that the true law of motion of m is given by the Ornstein–Uhlenbeck equation

$$\mathrm{d}m = -a^m m\ \mathrm{d}t + B^m\ \mathrm{d}Z^w + U^m\ \mathrm{d}Z^Q \tag{23}$$

with $a^m > 0$, i.e. m reverts to zero on average. Then (22) renders a perfectly intuitive result of the domestic currency (M) depreciating *relatively to the national asset return differential* when residents give up M in favor of I (m positive) and appreciating relatively to this differential when residents give up I and accumulate M (m negative).

Cross-border inter-dealer order flow is able to explain UTRP-deviations in both the symmetric and asymmetric information cases. Appendix B offers empirical examples of this. An important distinction of the asymmetric information case is that, differently from (23), the subjective drift parameter $\widehat{m}$ is no longer a martingale. This is why it is more natural to associate the persistence of non-stationary disparity term with the dealers' Bayesian information-acquisition at a finite speed, as described by (16) and (17).

Proposition 2 states that the residual disparity not explained by the combination of (22) (with the true value m instead of the estimate $\widehat{m}$) and (23) comes from the asymmetric information phenomena. The corresponding dynamic equilibrium in the forex depends on the dealers' learning process. One comes to three general conclusions about the properties of these asymmetric information equilibria:

1. At times when new information about the aggregate order flow ($\mathrm{d}M$) is being processed by the dealers, the disparity term is a non-stationary process. Its time-dependent parameters are generated by the evolving estimate precision in the course of learning the unobserved drifts of the fundamental variables (given by (17)). The fully specified model (16), (17), (20) for the risk premium is stationary. If the arrival of new information is a one-time event, the drift component of the disparity eventually converges to a constant.
2. The volatilities of both the exchange rate itself and the disparity term in the generalized UTRP equation should be higher under asymmetric information than in an FX market with fully informed dealers.
3. The arrival of a new order flow signal induces a change in a dealer's perceived covariance structure of the drifts of fundamental variables (matrix ω). Therefore, new information can have a permanent or, at least, very long-lived effect on the perceived dynamic of the aggregate order flow drift (process $\widehat{m}$) and, thereby, on the disparity/country premium level. The empirically observed revisions of the country premium can be explained in the logic of the model as a consequence of a revised interpretation of the order flow statistics monitored by the dealers.

Among the asymmetric information equilibria described in this section, there are many with sunspot properties. The reason is the self-fulfilling nature of beliefs about process m. The fact can be illustrated by an example where dealers have perfect information about the statistics of all components of Q except M.

4.2. Example: imprecise knowledge of the aggregate order flow only

The following one-dimensional example is technically analogous to the one in Genotte (1986) with the difference that Genotte lets the error term in the unobserved drift be dependent only on the observation noise (this special case would mean $B=0$ in our notation).

To treat this case, it is convenient to consider components Q' as exogenous parameters of the model, so that (redefined) processes $Q=M$ and $w=m$ have dimension 1. Covariance parameters $a^0, A, B, U, \omega, \mu = \Theta$ are now scalars. Let the true average trend of the inter-dealer order flow be a martingale of the (23) type, i.e. $a^0=0$ and $A=-a^m$. In such an economy, fully informed dealers would have generated a forex market with the UTRP satisfied exactly up to an additional random error term dependent on U and B.

Imperfectly informed dealers must learn the true value of m from the order flow observations. The innovation of their beliefs about m is described by the scalar s.d.e.

$$\mathrm{d}\,\widehat{m} = -\left[a^m\,\widehat{m} + \left(\frac{U}{\mu} + \frac{\omega}{\mu^2}\right)(\widehat{m} - m)\right]\,\mathrm{d}t + \frac{\omega}{\mu}\,\mathrm{d}Z^M \tag{24}$$

and the evolution of the precision of this subjective estimate by the scalar deterministic differential equation

$$\dot{\omega} = -\frac{\omega^2}{\mu^2} - 2\left(a^m + \frac{U}{\mu}\right)\omega + B^2.$$

The covariance parameter that satisfies this ordinary differential equation, converges to the stable steady state

$$\bar{\omega} = \mu^2\left\{\left[\left(a^m + \frac{U}{\mu}\right)^2 + \left(\frac{B}{\mu}\right)^2\right]^{1/2} - a^m - \frac{U}{\mu}\right\},$$

which never vanishes unless the diffusion parameter B in the s.d.e. for m is zero. That is, if there is an uncertainty factor driving m that cannot be diversified by observed risk factors, then ω never converges to zero and the dynamic of $\widehat{m}$ never fully converges to that of m. The coefficient b^m in (20) is equal to $(\omega/\mu)+U$.

Suppose that the true initial value of m is zero, but the subjective initial belief of the representative domestic dealer happens to be $\widehat{m}_0 > 0$. Then, from (24) we conclude that $\mathrm{d}\,\widehat{m}$ tends to be negative (the exchange rate movements are more downward sloping than what UTRP is prescribing). Eqs. (20) and (21) in this case tell us that at such times, the dealer's subjective shadow value of the exchange rate is more often higher than the true value, inducing him to initiate I-purchases in the inter-dealer market and, thereby, validate his beliefs. Put differently, the belief that everyone else purchases I, makes him purchase as well (the herding effect). The subsequent dynamic of P in

(20) after the initial "overshooting" move immediately after the formation of the prior belief $\widehat{m}_0 > 0$, is that of the downward adjustment of the exchange rate. In addition, on average, $|\mathrm{d}\,\widehat{m}| > |\mathrm{d}m|$, i.e. the partially informed dealer "overreacts" to the news about the current movement in the FX market, generating a higher volatility registered by outside observers.

The above example dealt with an extreme—and not particularly realistic—case of self-fulfilling beliefs in the inter-dealer market, while, possessing the maximum possible knowledge on other variables, the dealers had no incentives to correct their biased estimates of the aggregate inter-dealer order flow sufficiently. If the estimate update involves other macro state variables as well, one can expect that the sunspot effects in the m-variable will be mitigated. Specifically, errors in m and other unobserved drifts will partially offset each other. Similar effect in a Bayesian learning model in the context of interest rate estimation error was already noticed by Dothan and Feldman (1986).

5. Conclusion

The paper has developed a model of intertemporally optimizing FX dealers who use received order flow to improve their knowledge of fundamentals in a Bayesian manner. The model offered a contribution to the efforts at closing the existing gap between forex microstructure literature and traditional international macroeconomics, by identifying the common objects of study of both groups. The forex microstructure theory originally declared the intention to deal with the exchange rate formation in terms that could be recognized by financial market practitioners, as opposed to text-book macroeconomic lessons that are rarely reflected accurately by FX-traders. The problem is especially urgent in the eyes of a monetary authority whose very raison d'être is the presumed ability to implement a desired macroeconomic objective through actions taken in the money and FX markets. However, the best-known microstructure models existing to-date operate with conceptual shortcuts and information-theoretic constructions that make their messages even more, not less, distant from the dealer room language than standard propositions of classroom macroeconomics.[11] An additional problem is to derive from any of these models an empirically meaningful corollary testable on available data.

The paper has demonstrated that the long-term "macro" factors influencing the exchange rate and the short-term information dissemination about these factors among

[11] This is the author's experience from the dealing room conversations in a number of commercial and reserve banks. The conclusion is, not surprisingly, that a dealer is usually well versed in the uncovered interest rate parity argument (and even knows that it does not hold). He/she can also easily comment on the signal impact of released fresh data about inflation or GDP, on the exchange rate movements. It is, however, totally unrealistic to expect a dealer to analyze the separating versus pooling nature of equilibrium in a trader-specialist game of Easley and O'Hara (1987) or Glosten (1989). Scholars may use the latter models to *analyze* the actions of real dealerships, but not to *communicate* with them.

FX dealers and investors can be handled within a common continuous time portfolio optimization model. Properties of equilibria in this model account for

1. the uncovered asset return parity of the exchange rate and its generalizations,
2. the existence of variable stochastic spreads in a competitive multi-dealer environment,
3. concentration of individual dealer ask and bid quotes around a commonly observed clearing price,
4. dealer learning about changing fundamentals from private and inter-dealer trades, in the course of which, new information processing can cause deviations from the uncovered parity of total returns for the exchange rate,
5. permanent changes in the disparity constant (country premium) as a result of switching between self-fulfilling beliefs about the statistics of the inter-dealer order flow.

As was argued in Section 4, the origin of a change in self-fulfilling beliefs about the inter-dealer order flow can be the arrival of new information about the drift and the covariance matrix of the unobserved state process. This information is most likely to improve the currently available one, and only exceptionally, produce totally new pattern of co-movements between the fundamental characteristics of the economy, such as productivity, asset returns or the term structure of interest rates. Thus, if one excludes extreme overhauls of the long-term structural dependencies in the economy, the order flow effects described in the paper are unlikely to have a permanent impact on the exchange rate. On the other hand, this impact may be very long-lasting for the reasons of finite speed of learning by the agents, as expressed by the Kalman–Bucy filter equations featured in the text. On the empirical side, the persistence of order flow-caused effects is confirmed by sufficiently long episodes of deviation of the observed exchange rates from the UTRP.

Appendix A. The maximum principle solution of the dealer problem in continuous time

The following results utilize the adjoint equation techniques in stochastic optimal control of diffusions pioneered by Bismut (1976) and Hausmann (1981) and further developed in Peng (1990) and Cadenillas and Karatzas (1995).

The dealer's optimization problem discussed in Section 3 can be symbolically written down as

$$\max_{\ell} E\left[\int_0^T \mathrm{e}^{-[\vartheta]_0^t} U(X_t;\ell_t)\,\mathrm{d}t + \mathrm{e}^{-[\vartheta]_0^T} B(T,X_T)\right] \tag{A.1}$$

with respect to controls ℓ, subject to the state-transition equation

$$\mathrm{d}X = \mu(X,\ell)\,\mathrm{d}t + \sigma(X,\ell)\,\mathrm{d}Z, \tag{A.2}$$

the value X_0 of the state process at time $t=0$ given. The state process X is a vector with n components, and Z is a vector of d mutually independent standard Brownian motions. B is the so-called final bequest function. Its present value must possess a limit if the time horizon of the optimization problem is infinite ($T=\infty$). Finally, $[\vartheta]_t^s = \int_t^s \vartheta_\tau \, \mathrm{d}\tau$ is the discount factor between periods t and s.

Problem (A.1), (A.2) can be solved by forming the current value Hamiltonian

$$\mathsf{H}(t, X, \ell, \xi, \Xi) = U(X, \ell) + \xi \cdot \mu(X, \ell) - tr(\Xi \cdot \sigma(X, \ell)),$$

which is to be maximized with respect to ℓ_t. Here, ξ and Ξ are the *first-* and *second-order adjoint processes* (ξ is of the same dimension n as X and Ξ is an $n \times d$-matrix), with $\Xi = \xi \cdot D_x \sigma$ along the optimal path. When state X stands for asset holdings, the adjoint process ξ can be called the *shadow price* vector of the corresponding group of assets.

Let $[f, g]$ define the predictable co-variation of diffusion processes f and g, and put $d[f, g] = \langle f, g \rangle \, \mathrm{d}t$ (with the standard shorthand $\langle f \rangle$ for $\langle f, f \rangle$). Then the (first-order) adjoint process ξ satisfies the stochastic differential equation

$$\mathrm{d}\xi = \xi \cdot (\vartheta \mathbf{1}_n \, \mathrm{d}t - \mathrm{d}A + \langle A \rangle \, \mathrm{d}t) - D_X U \, \mathrm{d}t, \tag{A.3}$$

with the $n \times n$-matrix valued process A defined by

$$\mathrm{d}A = D_X \mu \, \mathrm{d}t + D_X \sigma \, \mathrm{d}Z, \qquad A_0 = \mathbf{1}_n.$$

The final condition $\xi_T = D_X B(T, X_T)$, or an appropriate transversality condition if $T=\infty$, must be added to (A.3). The adjoint process ξ can also be described as the X-gradient of the value function of problem (A.1), (A.2), provided the latter is differentiable.

In the investor-dealer problem of Section 2, the state-transition equation (A.2) is linear in the state variable x. Therefore, the coefficient matrix A and its quadratic variation $\langle A \rangle$ for this transition equation are easily seen to be equal to

$$A = \begin{bmatrix} \mathrm{d}r^0 & \mathrm{d}r^{\mathrm{d}} & 0 & 0 \\ 0 & -\mathrm{d}\alpha^{\mathrm{d}} & 0 & 0 \\ 0 & 0 & \mathrm{d}r^i & \mathrm{d}r^{\mathrm{f}} \\ 0 & 0 & 0 & -\mathrm{d}\alpha^{\mathrm{f}} \end{bmatrix},$$

$$\langle A \rangle = \begin{bmatrix} |I^{\mathrm{d}}|^2 & N^{\mathrm{d}} \cdot (I^{\mathrm{d}} - A^{\mathrm{d}}) & 0 & 0 \\ 0 & |A^{\mathrm{d}}|^2 & 0 & 0 \\ 0 & 0 & |I^{\mathrm{f}}|^2 & N^{\mathrm{f}} \cdot (I^{\mathrm{f}} - A^{\mathrm{f}}) \\ 0 & 0 & 0 & |A^{\mathrm{f}}|^2 \end{bmatrix}.$$

The previous two matrix expressions, if substituted into Eq. (A.3), render the adjoint equation system (6) of Section 2.

To derive the expression for the Hamiltonian of the dealer problem, one needs to calculate the terms containing the first- and the second-order adjoint process. First of all, observe that

$$\mu = \begin{bmatrix} x^0 i^{\mathrm{d}} + x^{\mathrm{d}} n^{\mathrm{d}} - \delta - X^{\mathrm{d}} k(v^{\mathrm{d}}) - Pk(v^i) + p^{\mathrm{a}} D(p^{\mathrm{a}}) - p^{\mathrm{b}} S(p^{\mathrm{b}}) + l \\ -x^{\mathrm{d}} a^{\mathrm{d}} + v^{\mathrm{d}} \\ -\dfrac{l}{P} + x^i i^{\mathrm{f}} + x^{\mathrm{f}} n^{\mathrm{f}} + v^i - X^{\mathrm{f}} k(v^{\mathrm{f}}) - D(p^{\mathrm{a}}) + S(p^{\mathrm{b}}) \\ -x^{\mathrm{f}} a^{\mathrm{f}} + v^{\mathrm{f}} \end{bmatrix}$$

and the part of the Hamiltonian containing the first order adjoint process is obtained by scalar multiplication of the above column vector by the row vector ξ. Further,

$$D_x \sigma = \begin{bmatrix} I^{\mathrm{d}} & N^{\mathrm{d}} & 0 & 0 \\ 0 & -A^{\mathrm{d}} & 0 & 0 \\ 0 & 0 & I^{\mathrm{f}} & N^{\mathrm{f}} \\ 0 & 0 & 0 & -A^{\mathrm{f}} \end{bmatrix}, \qquad \sigma = \begin{bmatrix} x^0 I^{\mathrm{d}} + \lambda + x^{\mathrm{d}} N^{\mathrm{d}} \\ -x^{\mathrm{d}} A^{\mathrm{d}} \\ -\dfrac{\lambda}{P} + x^i I^{\mathrm{f}} + x^{\mathrm{f}} N^{\mathrm{f}} \\ -x^{\mathrm{f}} A^{\mathrm{f}} \end{bmatrix}.$$

Therefore, the second-order adjoint process part of the Hamiltonian is equal to

$$-[\xi_0, \xi_{\mathrm{d}}, \xi_i, \xi_{\mathrm{f}}] \begin{bmatrix} x^0 |I^{\mathrm{d}}|^2 + I^{\mathrm{d}} \cdot \lambda + x^{\mathrm{d}} N^{\mathrm{d}} \cdot (I^{\mathrm{d}} - A^{\mathrm{d}}) \\ x^{\mathrm{d}} |A^{\mathrm{d}}|^2 \\ -\dfrac{I^{\mathrm{f}} \cdot \lambda}{P} + x^i |I^{\mathrm{f}}|^2 - \dfrac{x^{\mathrm{d}} N^{\mathrm{d}} \cdot \lambda}{P} + x^{\mathrm{f}} N^{\mathrm{f}} \cdot I^{\mathrm{f}} \\ x^{\mathrm{f}} |A^{\mathrm{f}}|^2 \end{bmatrix},$$

i.e. it does not contain the control variables. In short, maximizing the Hamiltonian with respect to the controls of the dealer problem is equivalent to maximizing the expression

$$\begin{aligned} &\xi_0 \{-X^{\mathrm{d}} k(v^{\mathrm{d}}) - Pk(v^i) + p^{\mathrm{a}} D(p^{\mathrm{a}}) - p^{\mathrm{b}} S(p^{\mathrm{b}})\} \\ &\quad + \xi_i \{v^i - X^{\mathrm{f}} k(v^{\mathrm{f}}) - D(p^{\mathrm{a}}) + S(p^{\mathrm{b}})\} - \xi_0 \delta + \xi_{\mathrm{d}} v^{\mathrm{d}} + \xi_{\mathrm{f}} v^{\mathrm{f}}. \end{aligned}$$

This maximization is fully described by the first-order conditions

$$\begin{aligned} &u'(\delta) = \xi_0, \\ &\xi_0 P k'(v^i) = \xi_i, \; \xi_0 X^{\mathrm{d}} k'(v^{\mathrm{d}}) = \xi_{\mathrm{d}}, \; \xi_i X^{\mathrm{f}} k'(v^{\mathrm{f}}) = \xi_{\mathrm{f}}, \\ &\xi_0 \lfloor D + p^{\mathrm{a}} D' \rfloor = \xi_i D', \qquad \xi_0 \lfloor S + p^{\mathrm{b}} S' \rfloor = \xi_i S' \end{aligned}$$

that we use in Section 2.2 of the main text.

Appendix B. Empirical evidence on the order flow-adjusted uncovered exchange rate parity: the Czech koruna case

Even if one came up with an appropriate discrete time version of the order flow-adjusted uncovered parity equation (20), it would be impossible for the general observer of the forex to estimate it in view of several unobservables present in that equation. These publicly unobserved variables include both the objective order flow and the subjective order flow estimation errors. Someone with access to the cross-border FX order flow data of the type employed by the model, could, at most, estimate a full information version of (20). Estimating the imperfect information uncovered parity by means of a filtering model in discrete time would be inappropriate, because the underlying dealer learning model in discrete time would have a state-space representation different from a simple discretization of (15)–(22). Developing the microfoundations for the discrete time FX inter-dealer model is outside the scope of the present paper. Therefore, when reporting here the empirical evidence on the GUTRP for the Czech koruna/euro exchange rate, we are only able to give limited results on

A. the symmetric information order flow-adjusted uncovered parity for the FX return, with the help of the daily cross-border CZK order flow data provided by the Czech National Bank (i.e. use as an explanatory variable a time series which, as the time it was being collected, was not available in real time but to the Czech monetary authority)
B. simulations of the imperfect information order flow-adjusted uncovered parity (the full-fledged version of (20)) with artificial state process parameters.

One can find periods when the UTRP for CZK/EUR returns even without the adjustment for the order flow is roughly satisfied. According to our model, this should be the case when there is neither a significant asymmetry between dealers nor any important information to be learned from the order flow. The estimations of a discrete-time counterpart of Eq. (9), as discussed in Derviz (2002), identify a number of such periods, some of them several months long (cf. Fig. 1).

An important technical issue to be solved both in A and B is finding the right model for residuals. As is well known, discrete time sampling of Itô equations typically leads to ARMA residuals instead of the desired i.i.d. ones. On the other hand, accommodation

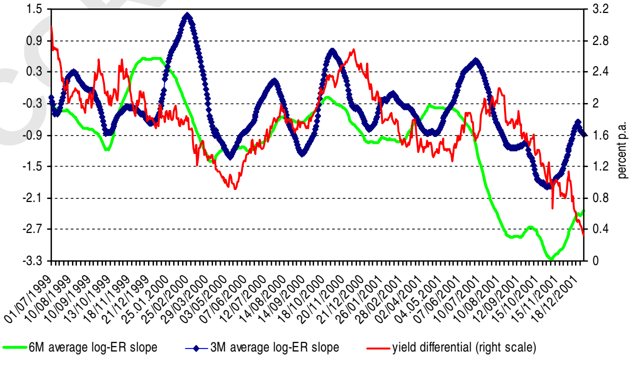

Fig. 1. Czech–German 10Y official bond yield differential, 3M- and 6M-moving average of the CZK/EUR exchange rate changes.

of ARMA terms in formal regressions often happens at the cost of reduced explanatory power by the original right-hand side variables. At present, the best solution seems to be preliminary filtering off the highest frequencies in the exchange rate, the asset return and the aggregate order flow series. These high frequencies correspond to diffusion terms in the continuous time parity equation.

One must also decide upon the time horizon over which the uncovered parity will be tested, and a smoothing/averaging procedure over the chosen horizon for the observed exchange rate movement series. As it turns out, the best-performing smoothing horizons vary between one and six months, depending on the analyzed currency pair and the historical period covered by the sample. Once the horizon is picked, the exact choice of averaging procedure does not play a decisive role. At the same time, there are episodes when UTRP seems to break down for fixed horizons, manifesting itself instead as a co-integration of the exchange rate *level* with the return differential. Differently from outright UTRP-violations, most of which can be explained within the present model, such episodes require the analysis of typical holding periods of given assets, and are left out of the present discussion.

As was explained in Section 3, long maturity government bonds proved to be the best instruments for UTRP analysis. The examples given below refer to ten-year bonds as the most widespread category to be found in both examined economies.

The main finding regarding Model A is a satisfactory performance of the full information order flow-adjusted UTRP for the exchange rate changes over 3M and 6M horizons for a number of sampling periods between 18 and 24 months long on daily data. The ex post moving slope of the nominal exchange rate logarithm 3 (6) months ahead of the current date is taken as the smoothing statistic mentioned above. The country premium (excess return) co-movements with the cross-border inter-bank order flow are illustrated in Fig. 2.

The cross-border inter-dealer CZK-EUR flows were obtained from the balance sheets of the FX dealer banks operating in the Czech koruna market (the Czech National Bank data). Fig. 2 shows the disparity term as the difference between the smoothed

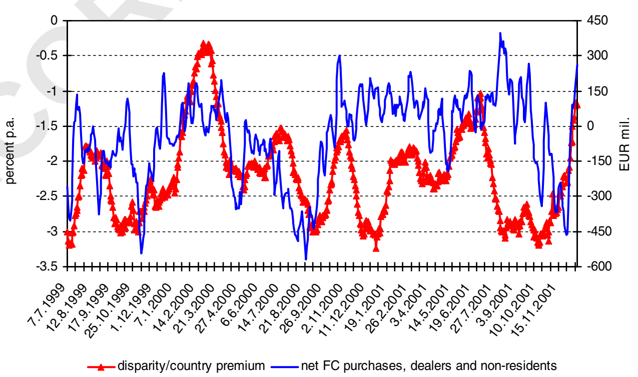

Fig. 2. Net cross-border order flow (foreign currency purchases, 5D-smoothed OF-series) and excess CZK/EUR returns.

CZK/EUR rate change (3M smoothing) and the Czech–German government bond differential. This time series is compared to the aggregate net euro-for-koruna purchases of the local Czech dealer banks from the foreign resident partners (clients and dealers; informal evidence suggests that those purchases/sales were predominantly initiated by non-resident counter-parties during the period in question). The latter is our proxy of m. We have regressed the disparity for 3M smoothed CZK/EUR return series on the said cross-border order flow proxy (HP-smoothed) with the ARMA(1,1)-residuals. The result is the following (t-statistics in parentheses):

$$\underset{}{\mathsf{DISP3M}} = \underset{(-5.40)}{-1.96} + \underset{(2.43)}{0.0007^{*}\mathsf{OFTREND}} + [\underset{(162.02)}{\mathsf{AR}(1) = 0.99}, \ \underset{(4.92)}{\mathsf{MA}(1) = 0.20}, \mathsf{BACKCAST} = 2].$$

The (adjusted) R^2 is 0.98, the regression s.e. is 0.078 and the DW statistic is 1.98.

One sees that even the simplest symmetric information UTRP stated in (9) can be roughly consistent with the data for as long as one year in a row. Longer periods become inappropriate for the UTRP-type reasoning, while the evolution of the disparity (country premium) term in multi-annual samples cannot be ignored. The data indicate that there occur regular episodes of the disparity term revision. Every such episode is eventually followed by the restored validity of UTRP, but it is impossible to make a single equation such as (9) comply with the whole sample at once.

The learning effects captured by (20) provide a theoretical explanation of the country premium revision. As a substitute for a rigorous analysis of a discrete time testable counterpart of the present model, I have simulated the country premium arising from (20) and (21), discrete random walks replacing diffusions, in two dimensions. Namely, I assumed that, beside the aggregate FX order flow, the second imperfectly observed fundamental (for the euro area-based FX dealer bank) is the return on Czech equity, $\mathrm{d}R^{\mathrm{f}} = i^{\mathrm{f}}\,\mathrm{d}t + \upsilon^{\mathrm{f}}\,\mathrm{d}Z^{Q}$. Thus, the two unobserved states of the simulated model are i^{f} and m, for which the true laws of motion are assumed to be of the mean-reverting form, m following (23) and i^{f}—the evolution law

$$\mathrm{d}i^{\mathrm{f}} = a^{i}(\mathrm{i} - i^{\mathrm{f}})\,\mathrm{d}t + B^{i}\,\mathrm{d}Z^{w} + U^{i}\,\mathrm{d}Z^{Q}.$$

Further, matrix U is assumed to be zero (i.e. the two drifts are unaffected by observation noises) and the conditional covariance matrix B generates (I) zero, (II) negative (III) positive correlation between the two states in the three variants simulated.

We compare the actual excess return, $\hat{\mathrm{d}}P - \mathrm{d}R^{\mathrm{d}} + \mathrm{d}R^{\mathrm{f}}$, to what is predicted by the model. Up to a constant, (cf. (22)), the excess return must be equal to coefficient κ (cf. the definition prior to Proposition 2) times

$$a^{m}m\,\mathrm{d}t + \chi^{mi}\varepsilon^{i}\,\mathrm{d}t + \xi^{mm}\varepsilon^{m}\,\mathrm{d}t - b^{m}\,\mathrm{d}Z^{Q}. \tag{B.1}$$

Here, ε^{i} is the estimation error for the return drift and ε^{m} is the estimation error for the order flow drift. Naturally, the full information Model A contains only the first of the four terms. The evolution of ε and χ is given by the two-dimensional versions of (19) and (17).

Fig. 3 plots the drift part of the simulated (all three variants mentioned above) and the actual error component (i.e. the second and the third terms in (B.1)) of the excess return/disparity for the analyzed sample. Evidently, the model in two dimensions is

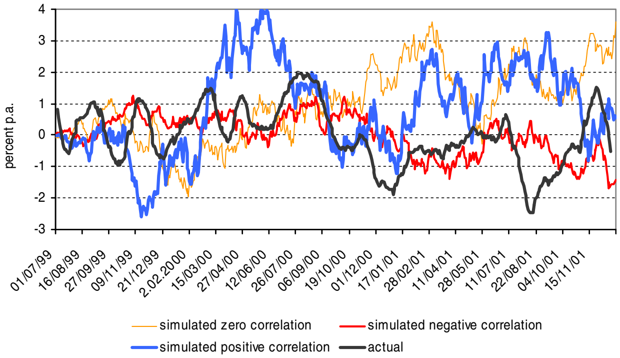

Fig. 3. Actual learning error drift in the uncovered parity equation and simulated errors for different correlations between two unobserved state variables (foreign asset return and the cross-border order flow).

able to produce deviations from UTRP in the magnitude comparable to the actual one and even bigger. Accidentally, the best imitation of reality is obtained by simulating a negatively conditionally correlated foreign return drift and order flow drift. However, one would need an estimation of all unobserved parameters in a properly specified discrete time learning model to make more substantiated statements about the dealer beliefs.

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