Collateral Shortages, Asset Price and Investment Volatility with Heterogeneous Beliefs*

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Abstract

The recent economic crisis highlights the role of financial markets in allowing economic agents, including prominent banks, to speculate on the future returns of different financial assets, such as mortgage-backed securities. This paper introduces a dynamic general equilibrium model with aggregate shocks, potentially incomplete markets and heterogeneous agents to investigate this role of financial markets. In addition to their risk aversion and endowments, agents differ in their beliefs about the future exogenous states (aggregate and idiosyncratic) of the economy. This difference in beliefs induces them to take large bets under frictionless complete financial markets, which enable agents to leverage their future wealth. Consequently, as hypothesized by Friedman (1953), under complete markets, agents with incorrect beliefs will eventually be driven out of the markets. In this case, they also have no influence on asset prices and real investment in the long run. In contrast, I show that under incomplete markets generated by collateral constraints, agents with heterogeneous (potentially incorrect) beliefs survive in the long run and their speculative activities permanently drive up asset price volatility and real investment volatility. I also show that collateral constraints are always binding even if the supply of collateral assets endogenously responds to their price. I use this framework to study the effects of different types of regulations and the distribution of endowments on leverage, asset price volatility and investment. Lastly, the analytical tools developed in this framework enable me to prove the existence of the "generalized" recursive equilibrium in Krusell and Smith (1998) with a finite number of agents.

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1 Introduction

The events leading to the financial crisis of 2007-2008 have highlighted the importance of belief heterogeneity and how financial markets create opportunities for agents with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds profited from the securities by short-selling them.

One reason for why there has been relatively little attention, in economic theory, paid to heterogeneity of beliefs and how these interact with financial markets is the *market selection hypothesis.* The hypothesis, originally formulated by Friedman (1953), claims that in the long run, there should be limited differences in beliefs because agents with incorrect beliefs will be taken advantage of and eventually be driven out the markets by those with the correct belief. Therefore, agents with incorrect beliefs will have no influence on economic activity in the long run. This hypothesis has recently been formalized and extended in recent work by Blume and Easley (2006) and Sandroni (2000). However these papers assume financial markets are complete and this assumption plays a central role in allowing agents to pledge all their wealth.

In this paper, I present a dynamic general equilibrium framework in which agents differ in their beliefs but markets are endogenously incomplete because of collateral constraints. Collateral constraints limit the extent to which agents can pledge their future wealth and ensure that agents with incorrect beliefs never lose so much as to be driven out of the market. Consequently, all agents, regardless of their beliefs, survive in the long run and continue to trade on the basis of their heterogeneous beliefs. This leads to additional leverage and asset price volatility (relative to a model with homogeneous beliefs or relative to the limit of the complete markets economy).

The framework introduced in this paper also enables a comprehensive study of how the survival of heterogeneous beliefs and the structure of financial markets affect aggregate capital accumulation in the long run. I also use this framework to study the impact of different types of regulations on welfare, asset price volatility and investment. The dynamic general equilibrium approach adopted here is central for many of these investigations. Since it permits the use of well-specified collateral constraints, it enables me to look at whether agents with incorrect beliefs will be eventually driven out of the market. It allows leverage and endogenous investment (supply of assets) and it enables me to characterize the effects of different types of policies on welfare and economic fluctuations.

The dynamic stochastic general equilibrium model with incomplete markets I present in this paper is not only useful for the analysis of the effects of heterogeneity in the survival of agents with different beliefs, but also includes well-known models as special cases, including recent models, such as those in Kubler and Schmedders (2003), Fostel and Geanakoplos (2008) and Geanakoplos (2009), as well as more classic models including those in Kiyotaki and Moore (1997) and Krusell and Smith (1998). For instance, this model allows for capital accumulation with adjustment costs in the same model in Krusell and Smith (1998) and shows the existence of a "generalized" recursive equilibrium. This equilibrium existence has been an open question in the literature. The generality is useful in making this framework eventually applicable to a range of questions on the interaction between financial markets, heterogeneity, aggregate capital accumulation and aggregate activity.

More specifically, I study an economy in dynamic general equilibrium with aggregate shocks and heterogeneous, infinitely-lived agents.[1] Aggregate shocks follow a Markov process. Consumers differ in terms of their beliefs on the transition matrix of the Markov process (for simplicity, these belief differences are never updated as there is no learning; in other words agents in this economy agree to disagree).[2] There is a unique final good used for consumption and investment, and several real and financial assets. There are two classes of real assets: one class of assets, which I call trees, are in fixed supply and the other class of assets are in elastic supply. Only assets in elastic supply can be produced using the final good. The total quantity of final good used in the production of real assets is the aggregate *real investment*. I assume that agents cannot short sell either type of assets. Assets in elastic supply are important to model real investment (asset production) and also to show that collateral constraints do not arise because of an artificially limited supply of assets.

Incomplete (financial) markets are introduced by assuming that all loans have to use financial assets as collateralized promises as in Geanakoplos and Zame (2002). Selling a financial asset is equivalent to borrowing and in this case, agents need to put up some real assets as collateral. Loans are non-recourse and there is no penalty for defaulting. Consequently, whenever the face value of the security is higher than the value of its collateral, the seller of the security can choose to default without further consequences. In this case, the security buyer seizes the collateral instead of receiving the face value of the security. I refer to equilibria of the economy with these financial assets as *incomplete markets equilibria* since the presence of collateral constrains introduces endogenous incomplete markets. Several key results involve the comparison of incomplete markets equilibria to the standard competitive equilibrium with complete markets.

Households (consumers) can differ in many aspects, such as risk-aversion and endowments. Most importantly, they differ in their beliefs concerning the transition matrix governing transitions across the exogenous states of the economy. Given the consumers' subjective expectations, they choose their consumption and real and financial asset holdings to maximize their intertemporal expected utility. In particular, the consumers' perceptions about the future value of each unit of real asset, including future rental prices and future resale value, determine the consumers' demand for new units of real assets. This demand, in turn, determines how many new units of real assets are produced. Hence, demand determines real investment in a fashion similar to the neoclassical Tobin's Q theory of investment.

The framework delivers several results. The first set of results, already mentioned above, is related to the survival of agents with incorrect beliefs. As in Blume and Easley (2006) and Sandroni (2000), with perfect, complete markets, in the long run, only agents with correct beliefs survive. Their consumption is bounded from below by a strictly positive number. Agents with incorrect beliefs see their consumption go to zero, as uncertainties realize. However, in any incomplete markets equilibrium, every agent survives because of

[1] Infinite horizon and infinitely-lived agents allow the use of sationary equilibria and the analysis of short run versus long run.

[2] Alternatively, one could assume that even though agents differ with respect to their initial beliefs, they partially update them. In this case, similar results would apply provided that the learning process is sufficiently slow (which will be the case when individuals start with relatively firm priors). In the paper, I also show how to incorporate learning into the framework.

no-default-penalty condition.[3] When agents lose their bets, they can simply walk away from their collateral while keeping their current and future endowments to come back and trade in the financial markets in the same period.[4] They cannot do so under complete markets because they can commit to delivering all their future endowments.[5]

More importantly, the survival or disappearance of agents with incorrect beliefs affects asset price volatility. To focus on asset price volatility, I consider economies with only fixed-supply real assets, i.e., Lucas trees. Under complete markets, agents with incorrect beliefs will eventually be driven out of the markets in the long run. The economies converge to economies with homogeneous beliefs, i.e., the correct beliefs. Markets completeness then implies that asset prices in these economies are independent of past realizations of aggregate shocks. In addition, asset prices are the net present discounted values of the dividend processes with appropriate discount factors. As a result, asset price volatility is proportional to the volatility of dividends if the aggregate endowment, or equivalently the equilibrium stochastic discount factor, only varies by a limited amount over time and across states. These properties no longer hold under incomplete markets. Given that agents with incorrect beliefs survive in the long run, they exert permanent influence on asset prices. Asset prices are not only determined by the aggregate shocks as in the complete markets case, but also by the evolution of the wealth distribution across agents. This also implies that asset prices are history-dependent as the realizations of past aggregate shocks affect the current wealth distribution. The additional dependence on the wealth distribution raises asset price volatility under incomplete markets above the volatility level under complete markets.

I establish this result more formally using a special case in which the aggregate endowment is constant and the dividend processes are I.I.D. Under complete markets, asset prices are asymptotically constant. Asset price volatility, therefore, goes to zero in the long run. In contrast, asset price volatility stays well above zero under incomplete markets as the wealth distribution changes constantly, and asset price depends on the wealth distribution. Although this example is extreme, numerical simulations show that its insight carries over to less special cases. In general, long-run asset price volatility is higher under incomplete markets than under complete markets.

The volatility comparison is different in the short run, however. Depending on the distribution of endowments, short run asset price volatility can be greater or smaller under complete or incomplete markets. This happens because the wealth distribution matters for asset prices under both complete markets and incomplete markets in the short run. This formulation also helps clarify the long-run volatility comparison. In the long run, under complete markets, the wealth distribution becomes degenerate as it concentrates only on agents with correct beliefs. In contrast, under incomplete markets, the wealth distribution remains non-degenerate in the long run and affects asset price volatility permanently. However, the wealth of agents with incorrect beliefs may remain low as they tend to lose their bets. Strikingly, under incomplete markets and when the set of actively traded financial assets is endogenous, the poorer the agents with incorrect beliefs are, the more they leverage to

[3] This is a sepcial case of no limited commitment.

[4] This is in contrast to the usual limited commitment literature where agents are assumed to be banned from trading in financial markets after their defaults.

[5] Even though the survival mechanism here is relatively simple (but also realistic), characterizing equilibrium variable such as asset prices and leverage in this environment is not an easy exercise.

buy assets. High leverage generates large fluctuations in their wealth, and as a consequence, large fluctuations in asset prices.

The results concerning volatility of asset prices also translate into volatility of real investment. Consequently, real investment under incomplete markets exhibits higher volatility than under complete markets. To illustrate this result, I choose a special case in which the aggregate endowment and productivity are constant over time. Under complete markets, as economies converge to economies with homogeneous beliefs, capital levels converge to their steady-state levels. Investments levels are therefore approximately constant; investment volatility is approximately zero. In contrast, under incomplete markets investment volatility remains strictly positive because it depends on the wealth distribution and the wealth distribution constantly changes as aggregate shocks hit the economies.

It is also useful to highlight the role of dynamic general equilibrium for some results mentioned above. In particular, the infinite horizon nature of the framework allows a comprehensive analysis of short-run and long-run behavior of asset price volatility. Such an analysis is not possible in finite horizon economies, including Geanakoplos's important study on the effects of heterogeneous beliefs on leverage and crises. For example, in page 35 of Geanakoplos (2009), he observes similar volatility as the economy moves from incomplete to complete markets. In my model, the first set of results described above shows that the similarity holds only in the short run. The long run dynamics of asset price volatility totally differs from complete to incomplete markets. In my model, the results are also based on insights in Blume and Easley (2006) and Sandroni (2000) regarding the disappearance of agents with incorrect beliefs. However, these authors do not focus on the effect of their disappearance on asset price or asset price volatility.

The dynamic general equilibrium of the economy also captures the "*debt-deflation*" channel as in Mendoza (2010), which models a small open economy. The economy in my paper also follows two different dynamics in different times, "normal business cycles" and "*debt-deflation cycles,*" depending on whether the collateral constraints are binding for any of the agents. In a debt-deflation cycle, the borrowing constraint binds. Then, when a bad shock hits the economy, the constrained agents are forced to liquidate their physical asset holdings. This *fire sale* of the physical assets reduces the price of these assets and tightens the constraints further and starting a vicious circle of falling asset prices. This paper shows that the debt-deflation channel still operates when we are in a closed-economy with endogenous interest rate, as opposed to exogenous interest rates as in Mendoza (2010) or Kocherlakota (2000).

The second set of results that follows from this framework concerns collateral shortages. I show that collateral constraints will eventually be binding for every agent in complete markets equilibrium provided that the face values of the financial assets with collateral span the complete set of state-contingent Arrow-Debreu securities. Intuitively, if this was not the case, the unconstrained asset holdings would imply arbitrarily low levels of consumption at some state of the world for every agent, contradicting the result that consumption is bounded from below. In other words, there are always shortages of collateral even if I allow for an elastic supply of collateral. This result sharply contrasts with those obtained when agents have homogenous beliefs but still have reasons to trade due to differences in endowments or utility functions. In these cases, if the economy has enough collateral, or can produce it, then collateral constraints may not bind and the complete markets allocation is achieved.

Heterogeneous beliefs, therefore, guarantee collateral shortages.

Another immediate implication of these results concerns Pareto inefficiency of incomplete markets equilibria. Incomplete markets equilibria are Pareto-suboptimal whenever agents strictly differ in their beliefs. This can be seen for the results that under complete markets equilibria, some agent's consumption will come arbitrarily close to zero while this never happens under incomplete markets. Intuitively, under complete markets agents pledged their future income, while collateral constraints put limits on such transactions. While allocations in which some agents experience very low levels of consumption may not be attractive according to some social welfare criteria, the equilibrium under complete markets is Pareto optimal under the subjective expectations of the agents. This result also implies that there is the possibility for Pareto improving regulations. However, given that this result is about unconstrained Pareto-efficiency, Pareto improving regulations might involve altering the incomplete markets structure.[6]

The above mentioned results are derived under the presumption that incomplete markets equilibria exist. However, establishing existence of incomplete markets equilibria is generally a challenging task. The third set of results establishes the existence of incomplete markets equilibria with a stationary structure. In their seminal paper, Geanakoplos and Zame (2002) show that, with collateral constraints, the standard existence proof a la Debreu (1959) applies. Kubler and Schmedders (2003) extend the existence proof to infinite horizon economies. I use the insights from these works to show the existence of incomplete markets equilibria in finite and infinite horizon economies with production and capital accumulation. Following Kubler and Schmedders (2003), I look for *Markov equilibria*, i.e., in which equilibrium prices and quantities depend only on the distribution of *normalized financial wealth* and the total quantities of assets with elastic supply. I show the existence of the equilibria under standard assumptions. I also develop an algorithm, based on the algorithm in Kubler and Schmedders (2003), to compute these equilibria. The same algorithm can be used to compute the complete markets equilibrium benchmark. One direct corollary of the existence theorem is that a "generalized" recursive equilibrium in Krusell and Smith (1998) with a finite number of agents exists.[7][8]

The fourth set of results attempts to answer some normative questions in this framework. Simple and extreme forms of financial regulations such as shutting down financial markets are not beneficial. Using the algorithm described above, I provide numerical results illustrating that these regulations fail to reduce asset price volatility and moreover they may also reduce the welfare of all agents because of the restrictions they impose on mutually beneficial trades. In particular, the intuition for the greater volatility under such regulations is that, when the collateral constraints are binding, regulations restrict the demand for assets. Therefore asset

[6] For a two-period version of my model, the concept of constrained Pareto-inefficiency due to Geanakoplos and Polemarchakis (1986) can be checked. In some cases, the economy can be constrained inefficient in this sense, due to pecuniary externalities.

[7] The calibrating term "generalized" comes from the fact that the mapping from current to future wealth distribution can be multi-valued as opposed to single-valued in Krusell and Smith (1998).

[8] Allowing for only an abitrary finite number of agents instead of a continuum of agents as in Krusell and Smith (1998) does not necessarily make the existence proof simpler given the aggregate shocks to the economy. Moreover, in the computational part of the paper, the authors also end up using the version of their model with a finite number of agents.

prices are lower than they are in unregulated economies. Agents, however, will eventually save their way out of the constrained regime, at which point, asset prices will become comparable to the unregulated levels. Movements between constrained and unconstrained regimes create high asset price volatility. These results suggest that Pareto-improving or volatility reducing regulations must be sophisticated, for example, incorporating state-dependent regulations.

This paper is related to the growing literature studying collateral constraints, started with a series of papers by John Geanakoplos. The dynamic analysis of incomplete markets is closely related to Kubler and Schmedders (2003). They pioneer the introduction of financial markets with collateral constraints into a dynamic general equilibrium model with aggregate shocks and heterogeneous agents. There are two main technical contributions of this paper relative to Kubler and Schmedders (2003). The first is to introduce heterogeneous beliefs using Radner (1972) rational expectations equilibrium concept: even though agents assign different probabilities to the aggregate shocks, they agree on the equilibrium outcomes, including prices and quantities, once a shock is realized. This rational expectations concept differs from the standard rational expectation concept, such as the one used in Lucas and Prescott (1971), in which subjective probabilities should coincide with the true conditional probabilities given all the available information. The second is to introduce capital accumulation and production in a tractable way. Capital accumulation or real investment is modelled through intermediate asset producers with convex adjustment costs that convert old units of assets into new units of assets using final good.[9] The analysis of efficiency is related to Kilenthong (2009) and Kilenthong and Townsend (2009). They examine a similar but static environment.

My paper is also related to the literature on the effect of heterogeneous beliefs on asset prices studied in Xiong and Yan (2009) and Cogley and Sargent (2008). These authors, however, consider only complete markets. The survival of irrational traders is studied Long, Shleifer, Summers, and Waldmann (1990) and Long, Shleifer, Summers, and Waldmann (1991) but they do not have a fully dynamic framework to study the long run survival of the traders.

Related to the survival of agents with incorrect beliefs, Coury and Sciubba (2005) and Beker and Chattopadhyay (2009) suggest a mechanism for agents' survival based on explicit debt constraints as in Magill and Quinzii (1994). These authors do not consider the effects of the agents' survival on asset prices. My framework is tractable enough for a simultaneous analysis of survival and its effects on asset prices and investment. Beker and Espino (2010) has a similar survival mechanism to mine based on the limited commitment framework in Alvarez and Jermann (2000). However, my approach to asset pricing is different because asset prices are computed explicitly as a function of wealth distribution. Moreover, my approach also allows a comprehensive study of asset-specific leverage. Kogan, Ross, Wang, and Westerfield (2006) explore yet another survival mechanism but use complete markets instead.

Simsek (2009b) also studies the effects of belief heterogeneity on asset prices. He assumes exogenous wealth distributions to investigate the question whether heterogeneous beliefs

[9]Lorenzoni and Walentin (2009) models capital accumulation with adjustment cost using used capital markets. Through asset producers, I assume markets for both used and new capital.

affect asset prices. In contrast, I study the effects of the endogenous wealth distribution on asset prices as well as asset price volatility. Simsek (2009a) focuses on consumption volatility. He shows that as markets become more complete, consumption becomes more volatile as agents can speculate more. My first set of results suggests that this comparative statics only holds in the short run. In the long run, the reverse statement holds due to market selection.

When capital accumulation is introduced, the model in this paper is a generalization of Krusell and Smith (1998) with financial markets and adjustment costs. In particular, the existence theorem 2 shows that a recursive equilibrium in Krusell and Smith (1998) exists. Krusell and Smith (1998) derives numerically such an equilibrium, but they do not formally show its existence. My paper is also related to Kiyotaki and Moore (1997), although I provide a microfoundation for the financial constraint (3) in their paper using the endogeneity of the set of actively traded financial assets.

At the time of the first draft of this paper in 2009, I was not aware of the recent papers that discuss some issues related to the ones I consider in this paper. Brumm, Grill, Kubler, and Schmedders (2011) show the importance of collateral requirements on asset price volatility in a similar model but with two trees and Epstein-Zin recursive preferences. Kubler and Schmedders (2011) show the importance of beliefs heterogeneity and wealth distribution on asset prices in a model with overlapping-generations.

The rest of the paper proceeds as follow. In section 2, I present the general model of an endowment economy and preliminary analysis of survival, asset price volatility under the complete markets benchmark as well as under incomplete markets. In section 3, I define and show the existence of incomplete markets equilibria under the form of Markov equilibria. In this section, I also prove important properties of Markov equilibria in this model. In section 4, I derive a general numerical algorithm to compute Markov and competitive equilibria. Section 5 focuses on assets in fixed supply with an example of only one asset to illustrate the ideas in sections 2 and 3. In Section 6, I present the most general model with capital accumulation, labor supply and production. Section 7 concludes with potential applications of the framework in this paper. Lengthy proofs and constructions are in Appendixes A-D.

2 General model

In this general model, there are heterogeneous agents who differ in their beliefs about the future streams of dividends. There are also different types of assets (for examples trees, land, housing and machines) that differ in their dividend process and their collateral value. For example, some of the assets can be used as collateral to borrow and others cannot. These assets are in fixed supplied as in Lucas (1978) in order to study the effects of belief heterogeneity on asset prices. In Section 6, I show that the model can also allow for assets in flexible supply and production in order to study the effects of belief heterogeneity on aggregate physical investment and aggregate economy activity. Assets in fixed supply presented in this section are special cases of assets in flexible supply with adjustment costs approaching infinity.

2.1 The endowment economy

Consider an endowment, a single consumption (final) good economy in infinite horizon with infinitely-lived agents. Time runs from $t = 0$ to ∞. There are H types of consumers

$$h \in \mathcal{H} = \{1, 2, \ldots, H\}$$

in the economy with a continuum of measure 1 of identical consumers in each type. These consumers might differ in many dimensions including per period utility function $U_h(c)$ (i.e., risk-aversion), discount rates β_h, and endowments of good e_h. The consumers might also differ in their initial endowment of physical assets, Lucas' trees,[10] paying off real dividends in terms of the consumption good. However, the most important dimension of heterogeneity is the heterogeneity in beliefs over the evolution of the exogenous state of the economy. There are S possible exogenous states (or equivalently shocks)

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}.$$

The states capture both idiosyncratic uncertainties, i.e., individual endowments, and aggregate uncertainties, i.e., the dividends from the physical assets.[11]

The evolution of the economy is captured by the past and current realizations of the shocks over time: $s^t = (s_0, s_1, \ldots, s_t)$ is the series of realizations of shocks up to time t. Notice that the space S can be chosen large enough to encompass both aggregate shocks, such as shocks to the aggregate dividends, and idiosyncratic shocks, such as individual endowment shocks. I assume that the shocks follow a Markov process with the transition probabilities $\pi(s, s')$. Moreover, $\mathcal{S}$ is an ergodic set.

Now, in contrast to the standard rational expectation literature, I assume that the agents do not have a perfect estimate of the transition matrix π. Each of them has their own estimate of the matrix, π^h.[12] However, these estimates are not very far from the truth, .i.e., there exist u and U strictly positive such that

$$u < \frac{\pi^h(s, s')}{\pi(s, s')} < U \quad \forall s, s' \in S \text{ and } h \in \mathcal{H}$$

where $\pi(s, s') = 0$ if and only if $\pi^h(s, s') = 0$ in which case let $\frac{\pi^h(s,s')}{\pi(s,s')} = 1$.This formulation allows for time varying belief heterogeneity as in He and Xiong (2011). In particular, agents might share the same beliefs in good states, $\pi^h(s, .) = \pi^{h'}(s, .)$, but their beliefs start diverging in bad states, $\pi^h(s, .) \neq \pi^{h'}(s, .)$.[13]

[10] See Lucas (1978)

[11] A state s can be a vector $s = (A, \epsilon_1, ..., \epsilon_H)$ where A consists of aggregate shocks and ϵ_h are idiosyncratic shocks.

[12] Learning can be easily incorporated into this framework as into this framework by allowing additional state variables which are the current beliefs of agents in the economy. As in Blume and Easley (2006) and Sandroni (2000), agents who learn slower will dissappear under complete markets. However they all survive under incomplete markets. The dynamics of asset prices describe here will corresponds to the short-run behavior of asset prices in the economy with learning.

[13] Simsek (2009b) shows in a static model that only the divergence in beliefs about bad states matters for asset prices.

Real Assets: As mentioned above, there are A physical assets $a \in \mathcal{A} = \{1, 2, \ldots, A\}$. These assets pay off state-dependent dividends $d_a(s)$ in final good. These assets can both be purchased and be used as collateral to borrow. This gives rise to the notion of leverage on each asset. The ex-dividend price of each unit of asset a in history s^t is denoted by $q_a(s^t)$. I assume that agents cannot short-sell these real assets. The supply K_a of asset a is given at the beginning of the economy, under the form of asset endowments to consumers.

Financial Assets: In each history s^t, there are also (collateralized) financial assets, $j \in \mathcal{J}$. Each financial asset j (or financial security) is characterized by a pair of vectors, promised payoffs and collateral requirement (b_j, k_j). Promises are a standard feature of financial asset similarly to Arrow's securities, i.e., asset j traded in history s^t promises next-period pay-off $b_j(s^{t+1}) = b_j(s_{t+1}) > 0$ in term of final good at the successor nodes $s^{t+1} = (s^t, s_{t+1})$. The non-standard feature is the collateral requirement. Agents can only sell the financial asset j if they hold shares of real assets as collateral. We associate j with an $A-$dimensional vector $k^j \geq 0$ of collateral requirements. If an agent sells one unit of security j, she is required to to hold k_a^j units of asset, $a \in \mathcal{A}$, as collateral. If an asset a is required as collateral for financial security $j \in \mathcal{J}$, the agent selling this financial asset is required to hold k_a^j in asset a.[14]

Since there are no penalties for default, a seller of the financial asset defaults at a node s^{t+1} whenever the total value of collateral assets falls below the promise at that state. By individual rationality, the actual pay-off of security j at node s^t is therefore always given by

$$f_{j,t+1}\left(s^{t+1}\right) = \min\left\{b_j\left(s_{t+1}\right), \sum_{a=1}^{A} k_a^j\left(q_a\left(s^{t+1}\right) + d_a\left(s^{t+1}\right)\right)\right\}. \tag{1}$$

Let $p_{j,t}(s^t)$ denote price of security j at node s^t.

Assumption 1 *Each financial asset requires at least a strictly positive collateral*

$$\min_{j \in \mathcal{J}} \max_{a \notin \mathcal{A}} k_a^j > 0$$

If a financial asset j requires no collateral then its effective pay-off, determined by (1) will be zero, it will be easy to show that in equilibrium its price, p_j, will be zero as well. We can thus ignore these financial assets.

Remark 1 *I will call the financial markets incomplete even if $\mathcal{J}$ is complete in the normal sense of complete spanning, i.e., $J \geq S$. The financial markets are endogenously incomplete due to the fact that agents are constrained in the positions they can take due to the collateral requirement and the fact that the total supply of collateral assets is finite. The collateral requirement is a special case of limited commitment, as if borrowers (sellers of the financial assets) have full commitment ability, they will not be required to put up any collateral to borrow.*

[14] Notice that, there are only one-period ahead financial assets. See He and Xiong (2011) for a motivation why longer term collateralized financial assets are not used in equilibrium.

Remark 2 *Consider the case in which a financial asset j requires only k_a^j units of asset a as collateral. When an agents purchase k_a^j units of asset a. At the same time she can pledge these units as collateral to borrow $p_{j,t}$,that is, selling the financial asset j at that price. So the leverage ratio associated to the transaction is*

$$L_j = \frac{k_a^j q_{a,t}}{k_a^j q_{a,t} - p_{j,t}}. \tag{2}$$

Even though there are many financial assets available, in equilibrium only some financial asset will be actively traded, which in turn determines which leverage levels prevail in the economy. In this sense, both asset price and leverage are simultaneously determined in equilibrium, as emphasized in Geanakoplos (2009).

Remark 3 *It will be shown in Appendix B that the set $\mathcal{J}$ of financial securities can depend on the history of the economy s^t, provided that there exists a $\underline{k} > 0$ such that*

$$\inf_{j \in \mathcal{J}_t} \max_{a \notin \mathcal{A}} k_a^j > \underline{k}$$

For example, $\mathcal{J}_t$ contains a financial security j that requires only an asset a as collateral with the collateral requirement depending on the history s^t, for example

$$k_{a,t}^j = \max_{s^{t+1}|s^t} \left\{ \frac{b_j(s_{t+1})}{q_a(s^{t+1}) + d_a(s^{t+1})} \right\}. \tag{3}$$

So in this example $k_{a,t}^j$ is the minimum collateral level that ensures no default, that is

$$f_{j,t+1}\left(s^{t+1}\right) = b_j(s_{t+1}) \;\; \forall s_{t+1} \in \mathcal{S}.$$

This constraint captures the situation in Kiyotaki and Moore (1997) in which agents can borrow only up to the minimum across future states of the future value of their land. With $S = 2$, and state non-contingent debts, i.e., $b_j(s_{t+1}) = b_j$, Geanakoplos (2009) argues that even if we allow for a wide range of collateral level, that is the unique collateral level that prevails in equilibrium (thus there is a unique level of leverage in each instance, according to the remark above). This statement for two future states still holds in this context of infinitely-lived agents as proved later in Section 5. However, this might not be true if we have more than two future states.

Consumers: Consumers are the most important actors in this economy. They can be hedge fund managers or banks' traders in financial markets.

In each state s^t, each consumer is endowed with $e_t^h = e^h(s_t)$ units of final good. I suppose there is a strictly positive lower bound on these endowments. This lower bound guarantees a lower bound on consumption if a consumer decides to default on all her debt and withdraw from the financial markets.

Assumption 2 $\min_{h,s} e_h(s) > \underline{e} > 0.$

For example, commercial banks receive deposits from their retail branches while these banks also have trading desks that trade independently in the financial markets. In Section 6, endowments can come in the form of labor endowments.

Consumer maximize their intertemporal expected utility with the per period utility functions $U_h(.) : \mathbf{R}^+ \longrightarrow \mathbf{R}$ that satisfy

Assumption 3 *U_h is concave and strictly increasing.*

Notice that I do not require U_h to be strictly concave. This assumption captures linear utility functions in Geanakoplos (2009) and Harrison and Kreps (1978).

Consumer h takes sequences of prices as given and solves[15]

$$\max \mathbf{E}_0^h \left[\sum_{t=0}^{\infty} \beta_h^t U_h \left(c_t^h \right) \right]$$

and in each history s^t, she is subject to the budget constraint

$$c_t^h + \sum_{a \in \mathcal{A}} q_{a,t} k_{a,t}^h + \sum_{j=1}^{J} p_{j,t} \phi_{j,t}^h \leq e_t^h + \sum_{j=1}^{J} f_{j,t} \phi_{j,t-1}^h + \sum_{a \in \mathcal{A}} \left(q_{a,t} + d_{a,t} \right) k_{a,t-1}^h \tag{4}$$

the collateral constraints

$$k_{a,t}^h + \sum_{j: \phi_{j,t} < 0} \phi_{j,t}^h k_a^j \geq 0 \ \forall a \in \mathcal{A} \tag{5}$$

One implicit condition from the assumption on utility functions is that consumptions are positive, i.e., $c_t^h \geq 0$. In the constraint (5), if the consumer does not use asset a as collateral to sell any financial security, then the constraint becomes the no-short sale constraint

$$k_{a,t}^h \geq 0. \tag{6}$$

The most important feature of the objective function is the superscript h in the expectation operator, $\mathbf{E}^h[.]$ that represents subjective beliefs when agents estimate their future expected utility. The expectation can also be re-written explicitly as

$$\sum_{t,s^t} P_h \left(s^t \right) \beta^t U_h \left(c_t^h \left(s^t \right) \right),$$

where $P_h(s^t)$ is the probability of history s^t under agent h's belief.

In the budget constraint (4), e_t^h is her endowment that can depend on the aggregate state s_t. Entering period t, the agent holds $k_{a,t-1}^h$ old units of real asset a and $\phi_{j,t-1}^h$ units of financial asset j. She can trade old units of real asset a at price $q_{a,t}$, buy new units of asset $k_{a,t}^h$ for time $t+1$ at price $q_{a,t}$. She can also buy and sell financial securities $\phi_{j,t}^h$ at price $p_{j,t}$. If she sells financial securities she is subject to collateral requirement (5).

[15] I also introduce the disutility of labor in the general existence proof in Appendix B in order to study employment in this environment. The existence of equilibria for finite horizon allows for labor choice decision. When we have strictly positive labor endowments, l^h, we can relax Assumption 2 on final-good endowments, e^h.

At first sight, the collateral constraint (5) does not have the usual property of financial constraints in the sense that higher asset prices do not seem to enable more borrowing. However, using the definition of the effective pay-off, $f_{j,t}$, in (1), we can see that this effective pay-off is increasing in the prices of physical assets, $q_{a,t+1}$. As a result, financial asset prices, $p_{j,t}$, are also increasing in physical asset prices. So borrowers can borrow more if $q_{a,t+1}$'s increase. Given that the borrowing constraints is effective through future asset prices, when we embed this channel in a production economy in Section 6, this constraint creates a feed-back mechanism from the financial sector to the real sector different from the one in Brunnermeier and Sannikov (2010).[16]

Even though the formulation and solution method presented below allow for heterogeneity in the discount rates, to focus on beliefs heterogeneity, I assume in this section that agents have the same discount factor.

Assumption 4 *Agents have the same discount factor $\beta_h = \beta \ \forall h \in \mathcal{H}$*

Equilibrium: In this environment, I define an incomplete markets equilibrium as follows

Definition 1 *An **incomplete markets equilibrium** for an economy with initial asset holdings*

$$\left\{k_{a,0}^h\right\}_{h\in\{1,2,\ldots,H\}}$$

and initial shock s_0 is a collection of consumption, real and financial asset holdings and prices in each history s^t,

$$\begin{array}{l}\left(\left\{c_t^h\left(s^t\right), k_{a,t}^h\left(s^t\right), \phi_{j,t}^h\left(s^t\right)\right\}_{h\in\{1,2,\ldots,H\}}\right.\\ \left.\left\{q_{a,t}\left(s^t\right)\right\}_{a\in\mathcal{A}}, \left\{p_{j,t}\left(s^t\right)\right\}_{j\in J_t(s^t)}\right)\end{array}$$

satisfying the following conditions:
i) Asset markets for each real asset a and for each financial asset j in each period clear:

$$\begin{aligned}\sum_{h=1}^{H} k_{a,t}^h\left(s^t\right) &= K_a\\ \sum_{h=1}^{H} \phi_{j,t}^h\left(s^t\right) &= 0.\end{aligned}$$

ii) For each consumer h, $\left\{c_t^h\left(s^t\right), k_{a,t}^h\left(s^t\right), \phi_{j,t}^h\left(s^t\right)\right\}$ solves the individual maximization problem subject to the budget constraint, (4), and the collateral constraint, (5).

Notice that by setting the set of financial securities J empty, we obtain a model with no financial markets in which agents are only allowed to trade in real assets, but they cannot short-sell these assets. This case corresponds to Lucas (1978)'s model with several trees and heterogeneous agents.

[16]Thanks to Markus Brunnermeir for pointing this out.

As a benchmark, I also study equilibria under complete financial markets. Consumers can borrow and lend freely by buying and selling Arrow-Debreu state contingent securities, only subject to the no-Ponzi condition.[17] In each node s^t, there are S financial securities. Financial security s delivers one unit of final good if state s happens at time $t+1$ and zero units otherwise. Let $p_{s,t}$ denote time t price and let $\phi^h_{s,t}$ denote consumer h's holding of this security. The budget constraint (4) of consumer h becomes

$$c^h_t + \sum_{a\in\mathcal{A}} q_{a,t}k^h_{a,t} + \sum_{s\in\mathcal{S}} p_{s,t}\phi^h_{s,t} \leq e^h_t + \phi^h_{s_t,t-1} + \sum_{a\in\mathcal{A}} (q_{a,t} + d_{a,t})\, k^h_{a,t-1} \tag{7}$$

Definition 2 *A* ***complete markets equilibrium*** *is defined similarly to an incomplete markets equilibrium except that each consumer solves her individual maximization problem subject to the budget constraint (7) and the no-Ponzi condition, instead of the collateral constraint (5).*

In the next subsection, I establish some properties of incomplete markets equilibrium. I compare each of these properties to the one of complete markets equilibrium.

2.2 General properties of incomplete and complete markets equilibria

Given the endowment economy, we can easily show that total supply of final good in each period is bounded by a constant $\overline{e}$. Indeed in each period, total supply of final good is bounded by

$$\overline{e} = \max_{s\in\mathcal{S}} \sum_{h\in\mathcal{H}} e^h(s) + \sum_{a\in\mathcal{A}} d_a(s)\, K_a \tag{8}$$

The first term of the right hand side is the total endowment of each individual. The second term is total dividends from the real assets. In incomplete or complete markets equilibria, the market clearing condition for final good implies that total consumption is bounded from above by $\overline{e}$. Given that consumption of every agent is always positive, consumption of each agent is bounded from above by $\overline{e}$, i.e.,

$$c_{h,t}\left(s^t\right) \leq \overline{e}\ \forall t, s^t. \tag{9}$$

Under the boundedness of total quantities of assets, we can show that in any incomplete markets equilibrium, consumption of consumers is bounded from below by a strictly positive constant $\underline{c}$. Two assumptions are important for this result. First, no-default-penalty allows consumers, at any moment in time, to walk away from their past debts and only lose their collateral assets. After defaulting, they can always keep their non-financial wealth (inequality (11) below). Second, increasingly large speculation by postponing current consumption is not an equilibrium strategy, because in equilibrium, consumption is bounded by $\overline{e}$ (see inequality 12 below). This assumption prevents agents from constantly postponing their consumption to buy assets. Formally, we have the following proposition

[17] No-Ponzi condition

$$\lim_{t\longrightarrow\infty} \sum_{s\in\mathcal{S}} \prod_{r=0}^{t-1} p_{s_{r+1}}(s^r)\, \phi^h_{s,t} \geq 0$$

Theorem 1 *Suppose that there exists a* $\underline{c}$ *such that*

$$U_h(\underline{c}) < \underbrace{\frac{1}{1-\beta}U_h(\underline{e})}_{endowment} - \underbrace{\frac{\beta}{1-\beta}U_h(\overline{e})}_{speculation}, \ \forall h \in \mathcal{H}, \tag{10}$$

where $\overline{e}$ *is defined in (8).Then in a incomplete markets equilibrium, consumption of each consumer in each history always exceeds* $\underline{c}$*.*

Proof. As in (9), we can find an upper bound for consumption of each consumer in each future period. Also, in each period, one of the feasible strategies of consumer h is to default on all her past debts at the only cost of losing all the collateral assets, but she can still at least consume her endowment from the current period on, therefore

$$U_h(c_{h,t}) + \mathbf{E}_t^h\left[\sum_{r=1}^{\infty}\beta^r U_h(c_{h,t+r})\right] \geq \frac{1}{1-\beta}U_h(\underline{e}). \tag{11}$$

Notice that in equilibrium, $\sum_h c_{h,t+r} \leq \overline{e}$ therefore $c_{h,t+r} \leq \overline{e}$. So

$$U_h(c_h) + \frac{\beta}{1-\beta}U_h(\overline{e}) \geq \frac{1}{1-\beta}U_h(\underline{e}) \tag{12}$$

This implies

$$U_h(c_h) \geq \frac{1}{1-\beta}U_h(\underline{e}) - \frac{\beta}{1-\beta}U_h(\overline{e}) > U_h(\underline{c})$$

■

Two remarks can be made here. First, condition (10) is satisfied immediately if

$$\lim_{c\longrightarrow 0} U_h(c) = -\infty,$$

for example, with log utility or CRRA utility with CRRA constant exceeds 1. Second, the lower bound of consumption, $\underline{c}$, is decreasing in $\overline{e}$. Therefore, the more there is of the total available final good, the more profitable speculative activities are and the more incentives consumers have to defer current consumption to engage in these activities.

This survival mechanism is similar to the one in Alvarez and Jermann (2000) and Beker and Espino (2010), in particular the first term in the right hand side of (10) captures the fact that the agents always have the option to default and go to autarky in which they only consume their endowment which exceeds $\underline{e}$ in each period. However, the survival mechanism in this paper also differs because, agents can always default on their promises and lose all their physical asset holdings, but they can always go back to financial markets to trade right after defaulting. The second term in the right hand side of (10) captures the fact that, the most the agents can lose after each trading period after coming back to the financial markets is $\overline{e}$. The agents are also likely to make losses after coming back if they hold incorrect beliefs.

One immediate corollary of Proposition 1 is that every consumer survives in equilibrium. Therefore, incomplete markets equilibrium differs from complete markets equilibrium when consumers differ in their beliefs. The proposition below shows that in a complete markets equilibrium, with strict difference in beliefs, consumption of certain consumers will come arbitrarily close to 0 at some history. The intuition for this result is that if an agent believes that the likelihood of a state is much smaller than what other agents believe, the agent will want to exchange his consumption in that state for consumption in other states. Complete markets allow her to do so but, in incomplete markets equilibrium, the collateral constraint limits the amount of consumption that she can sell in each state.

Proposition 1 *Suppose there are consumers with the correct belief and some consumers with incorrect beliefs. Moreover, the utility functions satisfy the Inada-condition*

$$\lim_{c \longrightarrow 0} U_h'(c) = +\infty \ \forall h \in \mathcal{H}. \tag{13}$$

Then, almost surely

$$\lim_{t \longrightarrow \infty} c_t^h = 0 \tag{14}$$

for each h such that π^h differs from π, that is h has incorrect belief.

Proof. In Appendix A, I show that almost surely, for agents h with incorrect beliefs

$$\lim_{t \longrightarrow \infty} \frac{P_h(s^t)}{P(s^t)} = 0.$$

This result can be used to prove (14). Indeed, from the first-order condition with respect to $\phi_{s,t}^h$, we have

$$p_{s_t}\left(s^{t-1}\right) U_h'\left(c_t^h\left(s^{t-1}\right)\right) = P_h\left(s^t | s^{t-1}\right) U_h'\left(c_t^h\left(s^t\right)\right).$$

Iterating backward for $r = t-2, ..., 0$ and multiplying side by side the same equation for each r, we obtain:

$$\begin{aligned} \left(\prod_{r=0}^{t-1} p_{s_{r+1}}\left(s^r\right)\right) U_h'\left(c_0^h\right) &= \prod_{r=0}^{t-1} P_h\left(s^{r+1} | s^r\right) U_h'\left(c_t^h\left(s^t\right)\right) \\ &= P_h\left(s^t\right) U_h'\left(c_t^h\left(s^t\right)\right). \end{aligned}$$

Therefore for every h, h'

$$\frac{U_h'\left(c_h\left(s^t\right)\right)}{U_{h'}'\left(c_{h'}\left(s^t\right)\right)} = \frac{P_{h'}\left(s^t | s_0\right) U_{h'}\left(c_{h'}(0)\right)}{P_h\left(s^t | s_0\right) U_h'\left(c_h\left(s_0\right)\right)} \tag{15}$$

From inequality (9), we have $U_{h'}'\left(c_{h'}\left(s^t\right)\right) > U_{h'}'(\bar{e})$, therefore

$$U_h'\left(c_h\left(s^t\right)\right) > \frac{P_{h'}\left(s^t | s_0\right) U_{h'}\left(c_{h'}(0)\right)}{P_h\left(s^t | s_0\right) U_h'\left(c_h\left(s_0\right)\right)} U_{h'}'(\bar{e}).$$

Pick h' among agents with the correct belief, that is $P_{h'} = P$, with probability 1

$$\lim_{t \longrightarrow \infty} \frac{P_{h'}(s^t)}{P_h(s^t)} = \infty.$$

Therefore

$$\lim_{t \longrightarrow \infty} U_h'\left(c_h\left(s^t\right)\right) = \infty$$

or equivalently, given the Inada condition (13), we obtain (14). ■

Corollary 1 *In a competitive equilibrium with complete markets, under the Inada condition (13), if agents strictly differ in their beliefs, then consumption of some agent approaches zero at some history of the world. Formally,*

$$\inf_{h,s^t} c_h\left(s^t\right) = 0.$$

Proof. If agents strictly differ in their beliefs, there exists a pair of agents h, h' such that $\pi^h \neq \pi^{h'}$. Applying Proposition 1 for $P = P_{h'}$ we have, $P_{h'}$ almost surely

$$\lim_{t\longrightarrow\infty} c_h\left(s^t\right) = 0$$

or

$$\inf_{s^t} c_h\left(s^t\right) = 0.$$

■

Notice that this result is rather surprising because even if agents are strictly risk-averse, they can also disappear over time if they have incorrect beliefs.[18] Because they can perfectly commit to pay their creditor using their future income. They can do so using short-term debts and keep rolling over their debts while using their present income to pay interests, which grows over time as their indebtedness grows. This result also sheds light on the survival mechanism in Theorem 1: agents have limited ability to pledge their future income, for example their labor income, to their creditors. As a result, they can always default and keep their future income. This limited commitment is even stronger in my setting than in Alvarez and Jermann (2000) and Beker and Espino (2010)[19] because after defaulting, agents can always come back and trade in the financial markets by buying new physical assets and then use these physical assets to sell financial assets.[20][21]This story is also different from the limited arbitrage story in Shleifer and Vishny (1997), where asset prices differ from their fundamental values because agents with correct beliefs hit their financial constraints before

[18]If the consumers with incorrect beliefs are risk-neutral, their consumption will go to zero immediately after a certain date.

[19]Following the language of Kehoe and Levine (2001), agents are debt-constrained in these two papers, while they are liquidity-constrained in my paper.

[20]In Chien and Lustig (2009), in equilibrium, borrowers are indifferent between defaulting and not defaulting due to the complete spanning properties of financial assets. In my model, when there is not complete spanning, there will be strict defaults in equilibrium. The numerical method in Section 4 can be used to solve for equilibria in Chien and Lustig (2009) as well. See Hopenhayn and Werning (2008) for a model in which equilibrium defaults happen due to stochastic outside options.

[21]The following story of the founder of Long Term Capital Management shows that traders in the financial markets often have limited commitment: John Meriwether worked as a bond trader at Salomon Brothers. At Salomon, Meriwether rose to become the head of the domestic fixed income arbitrage group in the early 1980s and vice-chairman of the company in 1988. In 1991, after Salomon was caught in a Treasury securities trading scandal, Meriwether decided to leave the company. Meriwether founded the Long-Term Capital Management hedge fund in Greenwich, Connecticut in 1994. Long-Term Capital Management spectacularly collapsed in 1998. A year after LTCM's collapse, in 1999, Meriwether founded JWM Partners LLC. The Greenwich, Connecticut hedge fund opened with $250 million under management in 1999 and by 2007 had approximately $3 billion. The Financial crisis of 2007-2009 badly battered Meriwether's firm. From September 2007 to February 2009, his main fund lost 44 percent. On July 8, 2009, Meriwether closed the fund.

being able to arbitrage away the price difference. Here, agents with incorrect beliefs hit their financial constraint more often and are protected by the constraint. Moreover, in the equilibria computed in Section 5.2, agents with the correct belief (the pessimists) never hit their borrowing constraint.

Due to different conclusions about agents' survival, the following corollary asserts that complete and incomplete markets allocations strictly differ when some agents strictly differ in their beliefs.

Corollary 2 *Suppose that conditions in Theorem 1 and Proposition 1 are satisfied. Then, an incomplete markets equilibrium never yields an allocation that can be supported by a complete markets equilibrium. By the Second Welfare Theorem, incomplete markets equilibrium allocations are Pareto-inefficient.*

Proof. In a incomplete markets equilibrium, consumptions are bounded away from 0, but in a complete markets equilibrium, consumptions of some agents will approach 0. Therefore, the two sets of allocations never intersect. ■

Using this corollary, we can formalize and show the shortages of collateral assets.

Proposition 2 (Collateral Shortages) *If financial markets are complete in terms of spanning, i.e., the set of the vectors of promises b_j has full rank. Then, for any given time t, with positive probability, the collateral constraints must be binding for some agent after time t.*[22][23]

Proof. We prove this corollary by contradiction. Suppose none of the collateral constraints are binding after a certain date. Then we can take the first-order condition with respect to the state-contingent securities. This leads to consumption of some agents to approach zero at infinity, as shown in the proof of Proposition 1. This contradicts the conclusion of Theorem 1 that consumption of each agent is bounded away from zero. ■

Notice that, as in Lucas (1978), agents can hold the physical assets for the risk-return and consumption-investment trade-offs. However, when their collateral constraints are binding, the agents use these assets solely as collateral to borrow. Consequently, I interpret the binding collateral constraints as collateral shortages. Araujo, Kubler, and Schommer (2009) argue that when there is enough collateral we might reach the Pareto optimal allocation. However, in the complete markets case, there will never be enough collateral.[24]

I also emphasize here the difference between belief heterogeneity and other forms of heterogeneity such as heterogeneity in endowments or in risk-aversion. The following proposition, in the same form as Theorem 5 in Geanakoplos and Zame (2007), shows that if consumers share the same belief and discount rate, there exist endowment profiles with which collateral equilibria attain the first-best allocations.

[22] It can also be shown that, at any moment of time, for *every* agent, the collateral constraint must be binding some time in future with positive probability.

[23] I also use the techniques developed in Beker and Chattopadhyay (2010) to show this result when $\mathcal{J}$ is non-empty but not complete.

[24] I show in Appendix B that even if the supply of these collateral assets is endogeneous, there will still be collateral shortages.

Proposition 3 *If consumers share the same belief and discount factor, there is an open set of endowment profiles with the properties that the competitive equilibrium can be supported by an incomplete markets equilibrium.*

Proof. We start with an allocation such that there is no trade in the complete markets equilibrium, then as we move to a neighborhood of that allocation, all trade can be collateralized. ■

Even though other dimensions of heterogeneity such as risk-aversion and endowments also create trading in financial markets, this proposition shows the importance of belief heterogeneity in driving up trading volume and resulting in binding collateral constraints.

Before moving to show the existence and study the properties of incomplete markets equilibria, we go back to the complete markets benchmark to study the behavior of asset price volatility. We will compare this volatility with volatility under incomplete markets and show that, in general, in the long run, asset price is more volatile in an incomplete markets equilibrium than it is in a complete markets equilibrium.

Proposition 4 *Suppose that there are some agents with the correct belief, in the complete markets equilibrium, asset prices are asymptotically independent of the past realizations of the aggregate shocks in the long run. Formally, there exists a set of asset prices $\overline{q}_a(s)$ as functions of the state of the economy such that, almost surely,*

$$\lim_{t\longrightarrow\infty}\left|q_a\left(s^t\right)-\overline{q}_a\left(s_t\right)\right|=0.$$

Proof. The detailed proof is in Appendix A. Sandroni (2000) shows that in the long run, only agents with the correct belief survive. Therefore, in the long run, the economy converges to the economy with homogeneous belief (rational expectation). In such an economy, given markets completeness, there exists a representative agent with an instantaneous utility function U_{Rep}, and her marginal utility evaluated at the total endowment determines asset prices

$$\begin{aligned}\overline{q}_a\left(s^t\right)U'_{\text{Rep}}\left(e\left(s_t\right)\right) &= \beta\mathbf{E}_t\left\{\left(\overline{q}_a\left(s^{t+1}\right)+d_a\left(s_{t+1}\right)\right)U'_{\text{Rep}}\left(e\left(s_{t+1}\right)\right)\right\} \\ &= \mathbf{E}_t\left\{\sum_{r=1}^{\infty}d_a\left(s_{t+r}\right)\beta^r U'_{\text{Rep}}\left(e\left(s_{t+r}\right)\right)\right\} \qquad (16)\end{aligned}$$

in which $e(s)$ is the aggregate endowment in the aggregate state s. We can see easily from this expression that $\overline{q}_a\left(s^t\right)$ is history-independent. ■

As the simulation of complete markets equilibrium in Section 5 shows, asset prices always depend on the wealth distribution of assets among agents, even so in the long run limit. However, as agents with incorrect beliefs vanish, the wealth distribution among agents with correct beliefs also converges to a constant distribution. So asset prices will depend only on the exogenous state of the economy.

Corollary 3 *When shocks are I.I.D. and the aggregate endowment is constant across states, with probability one, asset prices converge to constants in the long run.*

Proof. By assumption $e(s_{t+r}) = e$ for all t and r. So $U'_{\text{Rep}}(e)$ cancels out in both sides of (16). As a result

$$\begin{aligned}\overline{q}_a\left(s^t\right) &= \mathbf{E}_t\left\{\sum_{r=1}^{\infty} d_a\left(s_{t+r}\right)\beta^r\right\} \\ &= \frac{\beta}{1-\beta}\sum_{s\in\mathcal{S}}\pi(s)\,d_a(s).\end{aligned}$$

The second equality comes from the fact that shocks are I.I.D. ■

In contrast to complete markets equilibrium, in the next section I show that, in incomplete markets equilibrium, asset prices can be history-dependent, as past realizations of aggregate shocks affect the wealth distribution, which in turn affects asset prices.

One issue that might arise when one tries to interpret Proposition 4 is that, in some economy, there might not be any consumer whose belief coincides with the truth. For example, in Scheinkman and Xiong (2003), all agents can be wrong all the time, except they constantly switch from over-optimistic to over-pessimistic.[25] To avoid this issue, I again use the language in Blume and Easley (2006) and Sandroni (2000). I reformulate the results above using the subjective belief of each consumer.

Proposition 5 *Suppose that the Inada condition (13) is satisfied. Then each agent believes that:*
1) In complete markets equilibrium, only her and consumers sharing her belief survive in the long run. However, in incomplete markets equilibrium, everyone survives in the long run.
2) In complete markets equilibrium, asset prices are history-independent. However, in incomplete markets equilibrium, asset price can be history-dependent.

The properties in this section are established under the presumption that incomplete markets equilibria exist. The next section is devoted to show the existence of these equilibria with a stationary structure. The next two sections follow closely the organization in Kubler and Schmedders (2003). The first shows the existence and the second presents an algorithm to compute the equilibria.

3 Markov Equilibrium

3.1 The state space

I define the *normalized financial wealth*[26] of each agent by

$$\omega_t^h = \frac{\sum_a \left(q_{a,t}+d_{a,t}\right)k_{a,t-1}^h + \sum_j \phi_{j,t}^h f_{j,t-1}}{\sum_a \left(q_{a,t}+d_{a,t}\right)K_a}.$$

[25] Sandroni (2000) shows that if none of the agents has the correct beliefs, then only agents with beliefs closest to the truth survive, where distance is measured using entropy.

[26] Sometime I refer to this normalized financial wealth as simply financial wealth.

Let $\omega(s^t) = \left(\omega^1(s^t), ..., \omega^H(s^t)\right)$ denote the *normalized financial wealth distribution*. Then in equilibrium $\omega(s^t)$ always lies in the (H-1)-dimensional simplex Ω, i.e., $\omega^h \geq 0$ and $\sum_{h=1}^H \omega^h = 1$. ω^h's are positive because of the collateral constraint (5) that requires the value of each agent's asset holdings to exceed the liabilities from their past financial assets holdings. And the sum of ω^h equals 1 because of the asset market clearing and financial market clearing conditions.

I will show that, under conditions detailed in Subsection 3.3 below, there exists a Markov equilibrium over a compact state space. I look for an equilibrium in which equilibrium prices and allocations depend only on the states $(s_t, \omega_t) \in \mathcal{S} \times \Omega$.

Let the state space $\mathcal{X}$ consist of all exogenous and endogenous variables that occur in the economy at some node σ, i.e., $\mathcal{X} = \mathcal{S} \times \mathcal{V}$, where $\mathcal{S}$ is the finite set of exogenous shocks and $\mathcal{V}$ is the set of all possible endogenous variables.

In each node σ, an element $v(\sigma) \in \mathcal{V}$ includes: the normalized wealth distribution $(\omega_h(\sigma))_{h\in\mathcal{H}} \in \Omega$, together with consumers' decisions: consumption, H current consumption $\left(c^h(\sigma)\right)_{h\in\mathcal{H}}$, $HA + HJ$ real and financial asset holdings $\left(k_a^h(\sigma), \phi_j^h(\sigma)\right)_{h\in\mathcal{H}}$. It also includes A_1 prices of assets with fixed supply $(q_a(\sigma))_{a\in\mathcal{A}_1}$. Finally it includes J prices of the financial assets $(p_j(\sigma))_{j\in J}$. Therefore $\mathcal{V} = \Omega \times E \times \widehat{\mathcal{V}}$ with

$$\widehat{\mathcal{V}} = \mathbf{R}_+^H \times \mathbf{R}_+^{AH} \times \mathbf{R}^{JH} \times \mathbf{R}_+^A \times \mathbf{R}_+^J \tag{17}$$

the set of endogenous variables other than the wealth distribution.

Finally, let $\mathcal{X} \subset \mathcal{V}$ denote the set of vectors of all the endogenous variables that satisfy: 1) financial markets clear, 2) asset markets clear and 3) the budget constraints of consumers bind. Formally,

$$\begin{aligned} \sum_h \phi_j^h &= 0 \ \forall j, \\ \sum_h k_a^h &= K_a \ \forall a, \end{aligned}$$

and consumers' budget constraints hold with equality[27]

$$c^h = e^h + \omega^h (q + d) \cdot K - q \cdot k^h - p \cdot \phi^h. \tag{18}$$

Notice that binding budget constraints imply that the good market clears:

$$\sum_h c^h = \sum_h e^h + \sum_a d_a K_a.$$

3.2 Markov Equilibrium Definition

In order to define a Markov equilibrium, I use the following definition of expectation correspondence. Given a state $(s, v) \in \mathcal{X}$, the 'expectation correspondence'

[27] We use the vector product $a \cdot b = \sum_i a_i b_i$

$$g : \mathcal{X} \rightrightarrows \mathcal{V}^{\mathcal{S}}$$

describes all next period states that are consistent with market clearing and agents' first-order conditions. A vector of endogenous variables

$$\left(v_1^+, v_2^+, \ldots, v_S^+\right) \in g\left(x\right)$$

and $(s, v_s^+) \in \mathcal{X}$ for each $s \in \mathcal{S}$ if for all households $h \in \mathcal{H}$ the following conditions holds

a) For all $s \in \mathcal{S}$

$$\omega_s^{h+} = \frac{k^h \cdot (q_s^+ + d_s^+) + \sum_{j \in \mathcal{J}} \phi_j^h \min\left\{b_j\left(s\right), \sum_{a \in A} k_a^j\left(q^+ + d^+\right)\right\}}{\sum_a \left(q^+ + d^+\right) \cdot K}.$$

b) There exist multipliers μ_a^h corresponding to collateral constraints such that

$$\begin{aligned} 0 &= \mu_a^h - q_a U_h'\left(c^h\right) + \beta_h E^h\left\{\left(q_a^+ + d_a^+\right) U_h'\left(c^{h+}\right)\right\} && (19) \\ 0 &= \mu_a^h \left(k_a^h + \sum_{j \in \mathcal{J} : \phi_j^h < 0} k_a^j \phi_j^h\right) \\ 0 &\leq k_a^h + \sum_{j \in \mathcal{J} : \phi_j^h < 0} k_a^j \phi_j^h. \end{aligned}$$

c) Define $\phi_j^h\left(-\right) = \max\left(0, -\phi_j^h\right)$ and $\phi_j^h\left(+\right) = \max\left(0, \phi_j^h\right)$, there exist multipliers $\eta_j^h\left(+\right)$ and $\eta_j^h\left(-\right) \in \mathbf{R}^+$ such that

$$\begin{aligned} 0 &= \sum_{a \in \mathcal{A}} \mu_a^h k_a^j - p_j U_h'\left(c^h\right) + \beta^h E^h\left\{f_j^+ U_h'\left(c^{h+}\right)\right\} - \eta_j^h\left(-\right) && (20) \\ 0 &= -p_j U_h'\left(c^h\right) + \beta^h E^h\left\{f_j^+ U_h'\left(c^{h+}\right)\right\} + \eta_j^h\left(+\right) \\ 0 &= \phi_j^h\left(+\right) \eta_j^h\left(+\right) \\ 0 &= \phi_j^h\left(-\right) \eta_j^h\left(-\right). \end{aligned}$$

Definition 3 *A Markov equilibrium consists of a (non-empty valued) "policy correspondence," P, and a transition function F*

$$P : \mathcal{S} \times \Omega \times E \rightrightarrows \widehat{\mathcal{V}}$$

and

$$F : graph\left(P\right) \longrightarrow \mathcal{V}^{\mathcal{S}}$$

such that $graph\left(P\right) \subset \mathcal{X}$ and for all $x \in graph\left(P\right)$ and all $s \in \mathcal{S}$ we have $F\left(x\right) \subset g\left(x\right)$ and $\left(s, F_s\left(x\right)\right) \in graph\left(P\right)$.

Lemma 1 *A Markov equilibrium is an incomplete markets equilibrium according to Definition 1.*

Proof. This result is similar to the one in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994). We only need to show that the first order conditions as represented by Lagrange multipliers are sufficient to ensure the optimal solution of the consumers. This holds because the optimization each consumer faces is a convex maximization problem. ■

Before continue, let me briefly discuss asset prices in a Markov equilibrium.

We can rewrite that first-order condition with respect to asset holding (19) as

$$q_a U_h'\left(c^h\right) = \mu_a^h + \beta_h E^h \left\{\left(q_a^+ + d_a^+\right) U_h'\left(c^{h+}\right)\right\} \geq \beta_h E^h \left\{\left(q_a^+ + d_a^+\right) U_h'\left(c^{h+}\right)\right\}.$$

By re-iterating this inequality we obtain

$$q_{a,t} \geq E_t^h \left\{\sum_{r=1}^{\infty} \beta_h^r d_{t+r} \frac{U_h'\left(c_{t+r}^h\right)}{U_h'\left(c_t^h\right)}\right\}.$$

We have a strict inequality if there is a strict inequality $\mu_{a,t+r}^h > 0$ in the future. So the asset price is higher than the discounted value of the stream of its dividend because in future it can be sold to other agents, as in Harrison and Kreps (1978) or it can be used as collateral to borrow as in Fostel and Geanakoplos (2008). Proposition 2 shows some conditions under which collateral constraints will eventually be binding for every agent when they strictly differ in their belief. As a results, asset price is strictly higher than the discounted value of dividends.[28]

Equation (19) also shows that asset a will have collateral value when some $\mu_a^h > 0$, in addition to the asset's traditional pay-off value weighted at the appropriate discount factors. Unlike in Alvarez and Jermann (2000), attempts to find a pricing kernel which prices assets using their pay-off value might prove fruitless because assets with the same payoffs but different collateral values will have different prices. This point is also emphasized in Geanakoplos's' papers.

Equation (20) implies that

$$p_j = \frac{\sum_{a \in \mathcal{A}} \mu_a^h k_a^j}{U_h'\left(c^h\right)} + \beta_h E^h \left[f_j^+ \frac{U_h'\left(c^{h+}\right)}{U_h'\left(c^h\right)}\right].$$

As in Garleanu and Pedersen (2010), the price of financial asset j does not only depend on its promised payoffs in future states $\beta_h E^h \left[f_j^+ \frac{U_h'\left(c^{h+}\right)}{U_h'\left(c^h\right)}\right]$ but also on its collateral requirements $\frac{\sum_{a \in \mathcal{A}} \mu_a^h k_a^j}{U_h'\left(c^h\right)}$.

3.3 Existence and Properties of Markov equilibrium

The existence proof is similar to the ones in Kubler and Schmedders (2003) and Magill and Quinzii (1994).

We approximate the Markov equilibrium by a sequence of equilibria in finite horizon. There are three steps in the proof. First, using Kakutani fixed point theorem to prove

[28] We can also derive a formula for the equity premium that depends on the multilipliers μ similar to the equity premium formula in Mendoza (2010)

the existence proof of the truncated T-period economy. Second, show that all endogenous variables are bounded. And lastly, show that the limit as T goes to infinity is the equilibrium of the infinite horizon economy.

However, the most difficult part, including in the two related papers, is to prove the second step and this step involves the most of problem-specific economics intuitions. Basically showing that quantities are bounded is easy (especially with collateral constraint), but showing prices are bounded is more challenging. For example, what are the upper bounds of prices of long-lived assets? These prices may well exceed the current aggregate endowment. With physical investment in Section 6, we also have to bound the total supply of elastic-supply capital. I get around this difficulty by using the usual assumption in the neoclassical growth model: assuming capital depreciates and strictly concave production functions, then combining it with the artificial compact boxes trick in Debreu (1959).

Lemma 2 *Consider a finite horizon economy that last $T+1$ periods $t = 0, 1, \ldots, T$, identical to infinite horizon economy excepts consumers maximize the expected utility over $T+1$ periods*

$$\mathbf{E}_0^h \left[\sum_{t=0}^{T} \beta_h^t U_h \left(c_t^h \right) \right]$$

and in the last period $t = T$, there are no financial markets. In the first period, the budget constraint of agents h is

$$c_0 + \sum_{a \in \mathcal{A}} q_{a,0} k_{a,0} + \sum_{j=1}^{J} p_{j,0} \phi_{j,0} \leq e_0^h + \omega_0^h \sum_{a \in \mathcal{A}} \left(q_{a,0} + d_{a,0} \right) K_a \tag{21a}$$

instead of (4).An equilibrium exists given any initial condition

$$\left(s_0 \in \mathcal{S}, \omega_0 \in \Omega \right).$$

Instead of the usual budget constraints used in recursive equilibria, we use the condition that each consumer holds a share of the final total value of assets. This sharing rule can be implemented by assuming each agent h holds exactly ω_0^h share of each asset $a \in \mathcal{A}$.

Proof. The proof follows the steps in Debreu (1959) using Kakutani's fixed point theorem and is presented in the Appendix. However it uses a different definition of attainable sets. Indeed, in Definition 7 in the Appendix, negative excess demand (instead of zero excess demand as in the original text) is enough to guarantee the boundedness of the equilibrium allocations. In addition, I will also show that prices are strictly positive. ■

To prove that the Markov equilibrium exists, we need to first show that there exists a compact set in which finite horizon equilibria lie. We need the following assumption

Assumption 5 *There exist $\overline{c}, \underline{c} > 0$ such that*

$$\begin{aligned} & U_h \left(\underline{c} \right) + \max \left\{ \frac{\beta_h}{1 - \beta_h} U_h \left(\overline{e} \right), 0 \right\} \\ \leq \; & \min \left\{ \frac{1}{1 - \beta_h} \min_{s \in \mathcal{S}} U_h \left(\underline{e} \right), \min_{s \in \mathcal{S}} U_h \left(\underline{e} \right) \right\} \;\; \forall h \in \mathcal{H}. \end{aligned} \tag{22}$$

and

$$
\begin{aligned}
& U_h(\overline{c}) + \min\left\{\frac{\beta_h}{1-\beta_h} U_h(\underline{e}), 0\right\} \\
\geq\ & \max\left\{\frac{1}{1-\beta_h} U_h(\overline{e}), U_h(\overline{e})\right\}\ \forall h \in \mathcal{H}.
\end{aligned} \tag{23}
$$

The intuition for (22) is detailed in the proof of Theorem 1; it ensures a lower bound for consumption. (23) ensures that prices of real assets are bounded from above. Both inequalities are obviously satisfied by log utility.

Lemma 3 *Suppose 5 is satisfied then there is a compact set that contains the equilibrium endogenous variables constructed in Lemma 2 for every T and every initial condition lying inside the set.*

Proof. Appendix. ■

Theorem 2 *Under the same conditions, a Markov equilibrium exists.*

Proof. In Appendix B, I show the existence of Markov equilibria for a general model with capital accumulation as well. As in Kubler and Schmedders (2003), we extract a limit from the T-finite horizon equilibria. Lemma 3 guarantees that equilibrium prices and quantities are bounded as T goes to infinity. The proof use an alternative definition of attainable sets and also corrects several errors in the Appendix of Kubler and Schmedders (2003). ■

Corollary 4 *In a Markov equilibrium, every consumer survives.*

Proof. From the construction of the equilibrium $c^h(s^t) > \underline{c}$ for all h, t, s^t. ■

Corollary 5 *The Markov equilibrium is Pareto-inefficient if agents strictly differ in their beliefs.*

Proof. In Proposition 1, I show that under complete market, i.e. Pareto efficient allocation, consumption of some agents get arbitrarily close to zero in some history. Given the lower bound on consumption of each Markov equilibrium, an allocation corresponding to a Markov equilibrium is not a complete markets allocation. Therefore it is not a Pareto efficient allocation. ■

Proposition 6 *In contrast to the complete markets benchmark, in these Markov equilibria, asset prices can be history-dependent.*

Proof. The realization of aggregate shocks determines the evolution of the financial wealth distribution which is one factor that determines asset prices. ■

Proposition 7 *When the aggregate endowment is constant across states $s \in \mathcal{S}$, and shocks are I.I.D., long run asset price volatility is higher under incomplete markets than it is under complete markets.*

Proof. Corollary 4 show that, in the long run, under complete markets and the assumptions above, the economy converges to the one with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets and asset price $q_a(s^t)$ converges to prices independent of time and state. Hence, under complete markets, asset price volatility converges to zero in the long run. Under incomplete markets, asset price volatility remains well above zero as aggregate shocks constantly change the wealth distribution, which, in turn, changes asset prices and investment. ■

There are two components of asset price volatility. The first one comes from the volatility in the dividend process and aggregate endowment. The second one comes from wealth distribution, when agents strictly differ in their beliefs. However, the second component disappears under complete markets because only agents with correct beliefs survive in the long run. Whereas, under incomplete markets, this component persists. As a result, when we shut down the first component, asset price is more volatile under incomplete markets than it is under complete markets in the long run. In general, the same comparison holds or not depending on the long-run correlation between the first and the second volatility components under incomplete markets.

4 Numerical Method

In this section, I present an algorithm to compute Markov equilibria defined in the last section. This algorithm can also be used to compute complete markets equilibria and is presented in Appendix C.

4.1 General Algorithm

Suppose we need to find a function ρ defined over $S \times E$ on to a compact set $A \subset \mathbf{R}^N$, where S has a finite number of elements and E is convex and compact, and ρ satisfies the functional equation

$$\rho = f + T\rho$$

We then first discretize E by $\{e_1, e_2, \ldots, e_K\}$, and $\rho^n = (\rho_1^n, \rho_2^n, \ldots, \rho_S^n)$, each component is defined over $\{e_1, e_2, \ldots, e_K\}$. Let $\widetilde{\rho}_s^n$ be the extrapolation of ρ_s^n over E. Then

$$\rho_s^{n+1}(e_k) = \arg\min_{r \in A} \|r - \{f(e_k) + T\widetilde{\rho}_s^n(e_k)\}\| \tag{24}$$

If we have a fixed point $\rho^{n+1} = \rho^n$ and $f(e_k) + T\widetilde{\rho}_s^n(e_k) \in A$ then

$$\rho_s^n(e_k) = f(e_k) + T\widetilde{\rho}_s^n(e_k)$$

We present an implementation of this general algorithm to compute Markov Equilibria. We can also use the algorithm to compute competitive equilibria with complete markets. The state space in this case is the current consumption of each agent and the total supply of assets with elastic supply. The details are presented in the Appendix.

4.2 Algorithm to Compute Markov Equilibria

The construction of Markov equilibria in the last section also suggests an algorithm to compute them. The following algorithm is based on Kubler and Schmedders (2003). There is one important difference between the algorithm here and the original algorithm. The more important difference is that the future wealth distributions are included in the current mapping instead of solving for them using sub-fixed-point loops. This innovation reduces significantly the computing time, given that solving for a fixed-point is time consuming in MATLAB. Relatedly, in section 5, as we seek to find the set of actively traded financial assets, we can include the future asset price as one of the components of the function ρ.

We look for the following correspondence

$$\begin{aligned} \rho : \mathcal{S} \times \Omega &\longrightarrow \widehat{\mathcal{V}} \times \Omega^S \times \mathcal{L} \\ (s, \omega) &\longmapsto \left(\widehat{v}, \omega_s^+, \mu, \eta\right) \end{aligned} \tag{25}$$

$\widehat{v} \in \widehat{\mathcal{V}}$ is the set of endogenous variables excluding the wealth distribution and total capital, as defined in (17). $(\omega_s^+)_{s\in\mathcal{S}}$ are the wealth distributions in the S future states and μ, η are Lagrange multipliers as defined in subsection 3.2.

From a given continuous initial mapping $\rho^0 = (\rho_1^0, \rho_2^0, \ldots, \rho_S^0)$, we construct the sequence of mappings $\{\rho^n = (\rho_1^n, \rho_2^n, \ldots, \rho_S^n)\}_{n=0}^{\infty}$ by induction. Suppose we have obtained ρ^n, for each state variable (s, ω, K_a), we look for

$$\rho_s^{n+1}(\omega, K_a) = \left(\widehat{v}_{n+1}, \omega_{s,n+1}^+, \mu_{n+1}, \eta_{n+1}\right) \tag{26}$$

that solves the forward equations presented in the Appendix.

We construct the sequence $\{\rho^n\}_{n=0}^{\infty}$ on a finite discretization of $\mathcal{S} \times \Omega \times E$. So from ρ^n to ρ^{n+1}, we will have to extrapolate the values of ρ^n to outside the grid using extrapolation methods in MATLAB. Fixing a precision δ, the algorithm stops when $\|\rho^{n+1} - \rho^n\| < \delta$.

There are two important details in implementing this algorithm:

First, in order to calculate the $(n+1)$-th mapping ρ^{n+1} from the $n-$th mapping, we need to only keep track of the consumption decisions c^h and asset prices q_a and p_j. Even though other asset holding decisions and Lagrange multipliers might not be differentiable functions of the normalized financial wealth distribution, the consumption decisions and asset prices normally are.[29] Relatedly, when there are redundant assets, there might be multiple asset holdings that implement the same consumption policies and asset prices.[30]

Second, if we choose the initial mapping ρ^0 as an equilibrium of the $1-$period economy as in Subsection 3.3, then ρ^n corresponds to an equilibrium of the $(n+1)-$period economy. I follow this choice in computing an equilibrium of the 2-agent economy presented below.

5 Asset price volatility and leverage

This section uses the algorithm just described to compute incomplete and complete markets equilibria and study asset price and leverage. In order to focus on asset price, I only keep

[29] See Brumm and Grill (2010) for an algorithm with adapted grid points that deals directly with non-differentiabilities in the policy functions.

[30] See Cao, Chen, and Scott (2011) for such an example.

one real asset in fixed supply. Each financial asset corresponds to a leverage level. Suppose selling financial asset j requires k_j units of the fixed-supply asset as collateral and price of j is p_j. This operation is equivalent to buy k_j units of the real asset, at price q with p_j borrowed. Therefore, leverage as defined in Geanakoplos (2009), and in Remark 3 by the ratio between total value of the real asset over the down payment paid by the buyer:

$$L_j = \frac{k_j q}{k_j q - p_j} = \frac{q}{q - \frac{p_j}{k_j}}.$$

If in equilibrium, only one financial asset j is traded, the leverage level corresponding to the financial asset is called the leverage level of the economy.

To make the analysis as well as the numerical procedure simple, I allow for only one asset and two types of agents: optimists and pessimists, each in measure 1 of identical agents. The general framework in Section 2 allows for a wide range of financial assets with different promises and collateral requirements. However, given that the total quantity of collateral is exogenously bounded, in equilibrium, only certain financial assets are actively traded. I choose a specific setting based on Geanakoplos (2009), in which I can find exactly which assets are traded. The setting requires that promises are state-incontingent and in each exogenous state there are only two possible future exogenous states. The assets that are traded are the assets that allow maximum borrowing while keeping the payoff to lenders riskless. Endogenous financial assets interestingly generate the most volatility in the financial wealth distribution as agents borrow to the maximum and lose most of their financial wealth as they lose their bets but their wealth increases largely when they win. This volatility in the wealth distribution in turn feeds into asset price volatility.

An endogenous set of traded assets also implies endogenous leverage which has been the object of interest during the current financial crisis. In order to match the observed pattern of leverage, i.e., high in good states and low in bad states, I introduce the possibility for changing types of uncertainty from one aggregate state to others. This feature is introduced in Subsection 5.2.5.

To answer questions related to collateral requirements asked in the introduction, in Subsection 5.2.6, I allow regulators to control the sets of financial assets that can be traded. Given the restricted set, the endogenous active assets can still be determined. One special case is the extreme regulation that shuts down financial markets. There are surprising consequences of these regulations on the welfare of agents, on the equilibrium wealth distribution and on asset prices.

5.1 The model

There are two aggregate states $s = G$ or B and one single asset of which the dividend depends on the state s

$$d(G) > d(B).$$

The state follows an I.I.D. process, with the probability of high dividends, π, unknown to agents in this economy. The supply of the asset is exogenous and normalized to 1. Let $q(s^t)$ denote the ex-dividend price of the asset at each history $s^t = (s_0, s_1, \dots, s_t)$.

Financial Markets: At each history s^t, we consider the set of J of financial assets which promise state-independent payoffs next period. I normalize these promises to $b_j = 1$. Asset j also requires k_j units of the real asset as collateral. The effective pay-off is therefore

$$f_{j,t+1}\left(s^{t+1}\right) = \min\left\{1, k_j\left(q\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right)\right\}$$

Fostel and Geanakoplos (2008) and Geanakoplos (2009) argue that if we allow for the set J to be dense enough then in equilibrium the only financial asset traded in equilibrium is the one with the minimum collateral level $k^*\left(s^t\right)$ to avoid default:

$$k^*\left(s^t\right) = \max_{s^{t+1}|s^t}\left\{\frac{1}{q\left(s^{t+1}\right) + d\left(s^{t+1}\right)}\right\}.$$

This statement also applies for my general set up under the condition that in each history node, there are only two future aggregate states. The following proposition makes it clear.

Proposition 8 *Suppose in each history s^t, there are only two possible future aggregate states s_{t+1}. Given the set J, in any incomplete markets equilibrium, there is no more than one actively traded asset with collateral requirement less than or equal to $k^*\left(s^t\right)$. There is also no more than one actively traded asset with collateral requirement greater than or equal to $k^*\left(s^t\right)$.*

Proof. The proof of the first part requires an analysis of portfolio choice of the sellers of these securities and is detailed in Appendix D. For the second part, notice that all securities with collateral greater than or equal to $k^*\left(s^t\right)$ is riskless to the buyers, i.e. delivering 1 unit of final good regardless of the future states. Hence, these securities are sold at the same price. In addition, the sellers of the securities prefer selling securities with the least level collateral requirement to save their collateral. Therefore in equilibrium, only one security, with the collateral requirement the smallest above $k^*\left(s^t\right)$, is traded. ■

Imagine that the set J includes all collateral requirements $k_j \in \mathbf{R}^+, k_j > 0$.[31]. Proposition 8 says that only securities with collateral requirement exactly equals to $k^*\left(s^t\right)$ are traded in equilibrium. Therefore the only actively traded financial asset is riskless to its buyers. Let $p\left(s^t\right)$ denote the price of this financial asset. The endogenous interest rate is therefore

$$r\left(s^t\right) = \frac{1}{p\left(s^t\right)} - 1.$$

Consumers: There are two types of agents in this economy, optimists, O, and pessimists, P, each in measure one of identical agents. They have the same utility function

$$\sum_{t=0}^{\infty}\beta^t U\left(c_t\right), \tag{27}$$

and endowment e in each period. But they differ in their belief about the transition matrix of the aggregate state s. Suppose agent $h \in \{O, P\}$ estimates the probability of high dividends

[31] To apply the existence theorem 2 I need J to be finite. But we can think of J as a fine enough grid.

as $\pi_G^h = 1 - \pi_B^h$.We suppose $\pi_G^O > \pi_G^P$, i.e. optimists always think that good states are more likely than the pessimists think they are.

So each agent maximizes the inter-temporal utility (27) given their belief of the evolution of the aggregate state, and is subject to the budget constraint

$$c_t + q_t\theta_t + p_t\phi_t \leq e_t + (q_t + d_t)\,\theta_{t-1} + f_t\phi_{t-1}, \tag{28}$$

no short-sale constraint

$$\theta_t \geq 0, \tag{29}$$

and collateral constraint

$$\theta_t + \phi_t k^* \geq 0, \tag{30}$$

for each $h \in \{O, P\}$. At time t, each agent choose to buy θ_t units of real asset at price q_t and ϕ_t units of financial asset at price p_t. Moreover, Proposition 8 allows us to focus on only one level of collateral requirement k^*.

Given prices q and p, this program yields solution $c_t^h(s^t), \theta_t^h(s^t), \phi_t^h(s^t)$. In equilibrium prices $\{q_t(s^t)\}$ and $\{p_t(s^t)\}$ are such that asset and financial markets clear, i.e.,

$$\begin{aligned} \theta_t^O + \theta_t^P &= 1 \\ \phi_t^O + \phi_t^P &= 0 \end{aligned}$$

for each history s^t.

I define the financial wealth of each agent at the beginning of each period as

$$\omega_t^h = \frac{(q_t + d_t)\,\theta_{t-1}^h + f_t\phi_{t-1}^h}{q_t + d_t}.$$

Due to the collateral constraint, in equilibrium, ω_t^h must always be positive and

$$\omega_t^O + \omega_t^P = 1.$$

The pay-off relevant state space

$$\left\{\left(\omega_t^O, s_t\right) : \omega_t^O \in [0,1] \text{ and } s_t \in \{G, B\}\right\}$$

is compact. I look for Markov equilibria in which prices and allocations depend solely on that state. In Section 2, I show the existence of such a Markov equilibrium, and in Section 4, I develop an algorithm that computes such an equilibrium.

Again in a Markov equilibrium, there are two main factors that affect asset prices. The first factor is the (exogenous) aggregate state, s_t. The aggregate state affects prices through the endowments of agents. Because their endowments determine their consumption, and thus determine the marginal utility at which they evaluate the value of the asset. Aggregate states also affect asset prices through the evolution of future aggregate states, if these states are persistent. The second factor that I emphasize here is the (endogenous) financial wealth distribution, ω_t^O, as it affects the budget constraints of different agents, as shown in Figure 1 below. The financial wealth distribution may vary significantly, especially when some agents have limited non-financial wealth. The evolution of the financial wealth distribution, Figure 2 below, together with the evolution of the exogenous state of the economy, determines the dynamics of asset prices, including the price of the real asset $q(s^t)$ and the interest rate $r(s^t)$.

5.2 Numerical Results

Let

$$
\begin{aligned}
\beta &= 0.5 \\
d(G) &= 1 > d(B) = 0.2 \\
U(c) &= \log(c),
\end{aligned}
$$

and the beliefs are $\pi^O = 0.9 > \pi^P = 0.5$. I will vary the endowments of the optimists and the pessimists, e^O and e^P respectively, in different numerical exercises. As the numerical exercise serves for illustration purposes, I choose a low discount factor to speed up the convergence of the numerical procedure (less than 5 minutes for converge under $\beta = 0.5$).[32]To study the issue of survival and its effect on asset prices, I assume that the pessimists have the correct beliefs, i.e., $\pi = \pi^P = 0.5$. Thus the optimists are over-optimistics.

5.2.1 Asset Prices

Given that the main demand for the asset comes from the optimists, when their endowment is small, their demand is more elastic with respect to "normalized financial wealth". To investigate this relationship, I fix the endowment of the pessimists at

$$e^P = \begin{bmatrix} 10 & 10.8 \end{bmatrix}$$

and vary the endowment of the optimist

$$e^O = \begin{bmatrix} e & e \end{bmatrix}.$$

I keep the aggregate endowment constant by choosing the pessimists' endowment to be state dependent.

Incomplete Markets Equilibrium: I rewrite the budget constraint of the optimists (28) using normalized financial wealth, ω_t^O,

$$c_t + q_t\theta_t + p_t\phi_t \leq e^O + (q_t + d_t)\,\omega^O.$$

Therefore, their total wealth $e^O + (q_t + d_t)\,\omega^O$ affects their demand for the asset. If non-financial endowment e^O of the optimists is small relative to price of the asset, their demand for the asset is more elastic with respect to their financial wealth $(q_t + d_t)\,\omega^O$. I compute Markov equilibria for two values of the optimists' wealth $e = 1$ and 10. Figure 1 plots price of the asset as a function of the optimists' normalized financial wealth ω^O for the good state $s_t = G$. The figure for the bad state $s_t = B$ is similar.

The dashed line corresponds to the high "non-financial" wealth of the optimists: $e^O = 10$; the solid line corresponds to the low "non-financial" wealth of the optimists: $e^O = 1$. The figure shows that the elasticity of price with respect to ω^O increases as we reduce the non-financial wealth of the optimists from $e^O = 10$ to $e^O = 1$. This is the case because when e^O

[32]The MATLAB code for the paper works for higher discount factors. However, the convergence time is significantly higher for realistic values of β (for $\beta = 0.95$ it takes around 7 hours to converge). In Cao, Chen, and Scott (2011), we use the code to do a more careful calibration exercise.

is sufficiently high, the optimists can satisfy most of their demand for the asset using their endowment instead of their financial wealth. Interest rate r is also endogenously determined in these economies, however most of the time it hovers around the common discount factors of the two agents, i.e. $r(s^t) \approx \frac{1}{\beta} - 1$.

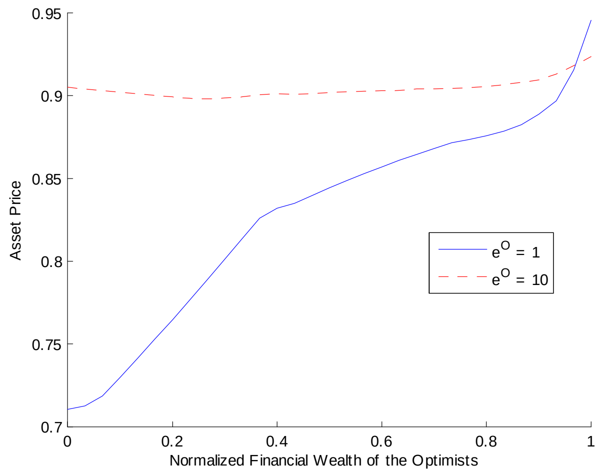


Figure 1: Asset Price Under Incomplete Markets

In order to study the dynamics of asset price, we need to combine the fact that asset price as a function of the normalized financial wealth shown in Figure 1 with the evolution of the exogenous state and the evolution of the "normalized financial wealth" distribution, ω_t^O. Figure 2 shows the evolution of ω_t^O when the non-financial endowment of the optimists is relatively small with respect to the price of the asset: $e^O = 1$. The left panel corresponds to the current good state $s_t = G$, and the right panel corresponds to the current bad state $s_t = B$. The solid lines represent next period normalized wealth of the optimists as a function of the current normalized wealth, if good shock realizes next period. The dashed lines represent the same function when the bad shock realize next period. I also plot the 45 degree lines for comparison. This figure shows that, in general, good shocks tend to increase and bad shocks tend to decrease the normalized wealth of the optimists. This is because the optimists bet more for the good state to happen (buy borrowing collateralized and investing into the real asset).

We can also think of normalized wealth as the fraction of the trees that the optimists hold. When the current state is good, and the fraction is high, the optimists will get a lot of dividends from their tree holdings, due to consumption smoothing they will not consume all the dividends, but will use some part of the dividends to buy new trees, so we see that on the left panel, the tree holding of the optimists normally increases at high ω^O. Similarly when the current state is bad, and the fraction is high, the optimists will sell off some of their tree holdings to smooth consumption. As a result, we see on the right panel that the tree holding of the optimists normally decreases at high ω^O.

When ω^O is close to zero, the optimists are highly leveraged to buy the asset. If a bad shocks hits in the next period, they have to sell off their asset holdings to pay off their debts. Their next period financial wealth plummets and contributes to the fall in asset price. Consequently, we can see in Figure 1, in the blue line, which corresponds to $e^O = 1$, the slope of the price function is steeper for $\omega^O < 0.4$. This also corresponds to the region in which the borrowing constraint binds. Then when a bad shock hits the economy, that is $s_{t+1} = B$, the optimists are forced to liquidate their physical asset holdings. This *fire sale* of the physical asset reduces the price of capital and tighten the constraints further and setting the vicious circle of falling asset price. Notice that, this channel also explains a faller slope at the lowest level of financial wealth of the optimists as they hold less of the physical asset, the asset firesale has less bite. This dynamics of asset price under borrowing constraint corresponds to the "*debt-deflation*" channel in a small-open economy in Mendoza (2010). This example show that the channel still operates when we are in a closed-economy with endogenous interest rate, $r(s^t)$ as opposed to exogenous interest rates in Mendoza (2010) or in Kocherlakota (2000).

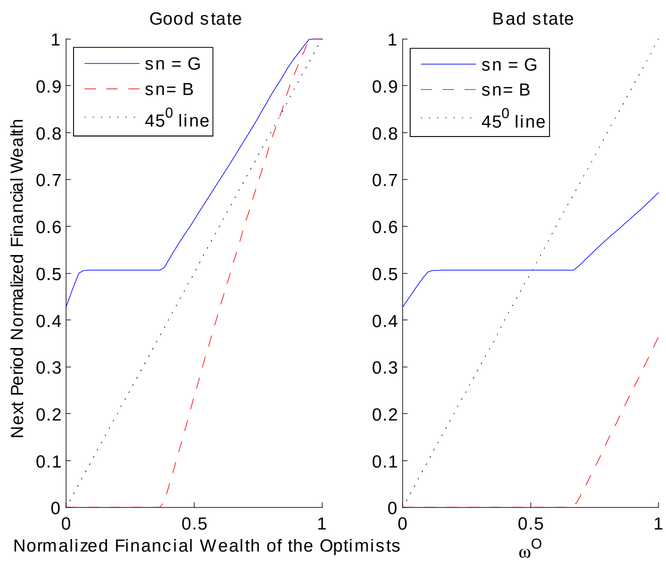


Figure 2: Dynamics of Wealth Distribution under Incomplete Markets

Complete Markets Equilibrium: In a complete markets equilibrium, as shown in Appendix C, Remark 4, the state variable is the consumption of the optimists. However, there is a one-to-one mapping from this state variable to a more meaningful state variable which is the relative wealth of the optimists. Given that markets are complete, wealth of each consumer is defined as the current value of her current and future stream consumption

$$V_t^h = \sum_{r=0}^{\infty} p_t\left(s^{t+r}\right) c_{t+r}^h\left(s^{t+r}\right),$$

where $p_t(s^{t+r})$ denotes the time t Arrow-Debreu price for a claim to a unit of consumption at date $t + r$ and sate s^{t+r}. Let

$$\widehat{\omega}_t^O = \frac{V_t^O}{V_t^O + V_t^P}$$

denote the relative wealth of the optimists with respect to the total wealth. Similar to the incomplete markets equilibrium, this variable determines asset price and constantly changes as aggregate shocks hit the economy. Figure 3 depicts the relationship between asset price and relative wealth[33]:

$$q(\widehat{\omega}) = \sum_{h\in\{O,P\}} \widehat{\omega}_h \frac{\beta}{1-\beta}\left(\pi^h d(G) + \left(1-\pi^h\right) d(B)\right). \tag{31}$$

This figure is the counterpart of Figure 1 for complete markets.[34]

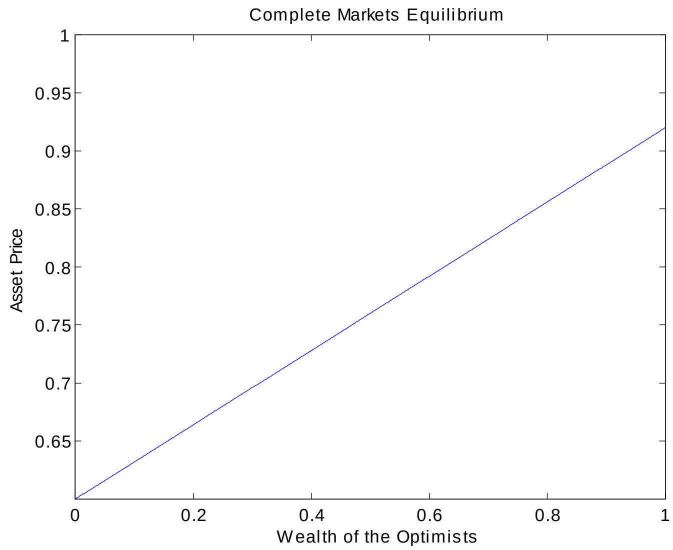


Figure 3: Asset Price under Complete Markets

Notice that at two extreme $\widehat{\omega}_t^O = 0$ (on the left of Figure 3) or 1 (on the right of Figure 3), we go back to the representative agent economy in which there are either only the optimists or the pessimists.[35]

In this special case where the aggregate endowments are constant across states and shocks are I.I.D., applying Corollary 3, we have in the long run, with probability 1, the price of the real asset, $q(s^t)$ converges to a constant

$$q = \frac{\beta}{1-\beta}\left(P(G)\,d(G) + P(B)\,d(B)\right), \tag{32}$$

[33]The derivation of this formula is too long to include in the paper and is available upon request.

[34]Unlike under incomple markets, under complete markets the asset price function does not depend on the exogenous state s_t due to the I.I.D assumption and constant aggregate endowment.

[35]When $\widehat{\omega}_t^O = 0$, asset price is the discounted value of average dividends evaluated at the pessimists' belief

$$q^P = \frac{\beta}{1-\beta}\left(\pi^P d(G) + \left(1-\pi^P\right) d(B)\right),$$

which is smaller than when $\widehat{\omega}_t^O = 1$, where asset price is the discounted value of average dividends evaluated at the optimists' belief

$$q^O = \frac{\beta}{1-\beta}\left(\pi^O d(G) + \left(1-\pi^O\right) d(B)\right) > q^P.$$

i.e., asset price volatility decreases to zero in the long run. Another way to see this convergence, is to notice that the wealth distribution $(\widehat{\omega}_O, \widehat{\omega}_P)$ converges to $(0,1)$; thus according to (31), q_t converges to $q\,(0)$ given by (32).

In the short-run, however, the wealth distribution constantly changes as shocks hit the economy. Figure 4 depicts the evolution of the relative wealth distribution that determines the evolution of asset price under complete markets. This figure is the counterpart of Figure 2 under complete markets. Given that the aggregate endowment is constant, the transition of the wealth distribution does not depend on current aggregate state, unlike under incomplete markets. The optimists buy more Arrow-Debreu assets that deliver in the good future states and buy less Arrow-Debreu assets that deliver in bad future states. Therefore, when a good shock hits, the relative wealth of the optimists increases (solid line) and vice versa when a bad shock hits (dashed line). Notice that, as opposed to the Figure 2, $\widehat{\omega}_O = 0$ and $\widehat{\omega}_O = 1$ are two absorbing states. So the optimists disappear under complete markets but not under incomplete markets.

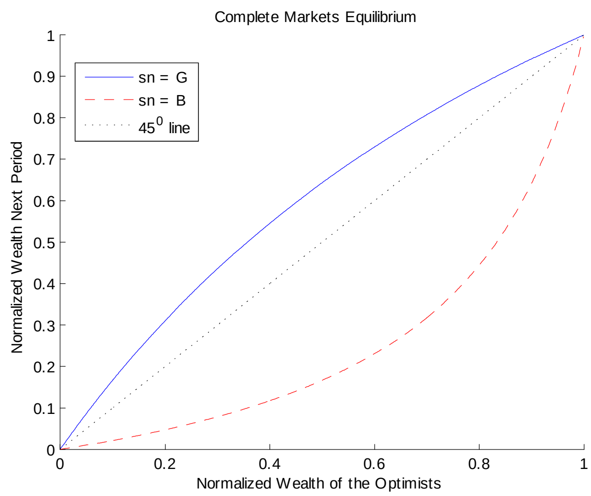


Figure 4: Dynamics of Wealth Distribution under Complete Markets

5.2.2 Survival under incomplete markets

Figure 5 shows a realization of the financial wealth of the optimist starting at $\omega^O = 0.5$. The optimists always lever up to buy the real asset and often they will lose all their asset holdings (selling off their asset holdings to pay off their debt), in which case, their financial wealth revert to zero. However, they can always use their non-financial endowment to come back to the financial markets and invest in the real asset again (leveraged). Sometime they are lucky, that is, when the asset pays high dividends and its price appreciates, their financial wealth can increase rapidly. Given this dynamics, there exists a non-degenerate stationary financial wealth distribution of the optimists, shown in Figure 6. The spikes of the distribution (including the one at 0) shows that the financial wealth of the optimists often reverts to 0 and after that the optimists come back to the financial markets with certain levels of financial wealth. This is in contrast with the results from the same simulation exercise for complete markets, where with probability 1 the wealth of the optimists will go to zero in the long run, thus the stationary distribution of the wealth of the optimists will be a degenerate mass at 0.

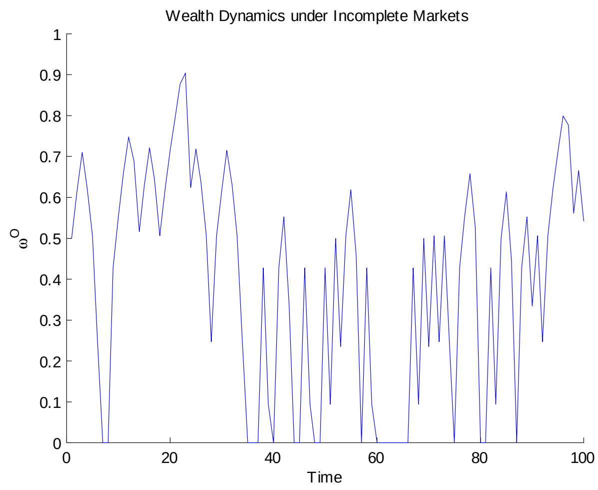


Figure 5: A Sample Path of Wealth Distribution

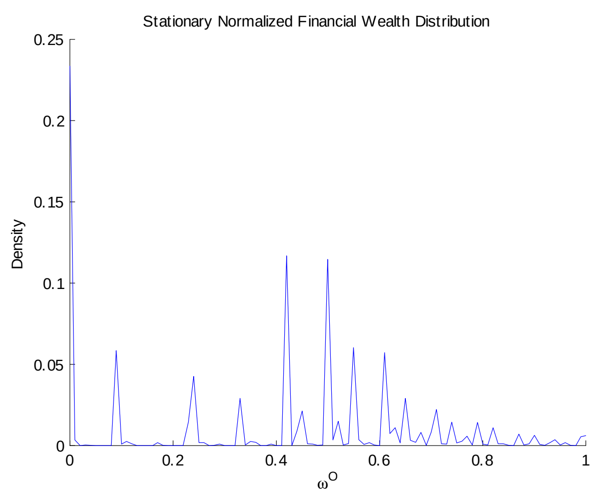


Figure 6: Stationary Normalized Wealth Distribution

5.2.3 Asset Price Volatility

We compare asset prices and asset price volatility of the Markov equilibrium under the complete markets benchmark. Consider first what happens with complete markets: Asset price does depend on the wealth distribution $\widehat{\omega}_t^O$ and its evolution. However, in the long run $\widehat{\omega}_t^O$ converges to 0 or to 1 depending on whether the pessimists or the optimists hold the correct belief. Therefore, in the long run, asset price only depends on the exogenous states. But given the constant aggregate endowment across the exogenous states and I.I.D.

shocks, asset price converges to a constant and asset price volatility converges to zero in the long run. In the short run, as the wealth distribution also moves over time under complete markets, asset price volatility might be very large.

In the case of Markov equilibrium under incomplete markets, however, consumers with incorrect beliefs, the optimists, are protected by the collateral constraint. They always survive in equilibrium, and constantly speculate on asset prices. First, asset price is not only state dependent but also depends on the wealth of the optimists. Second, their wealth undergoes large swings as they lose or win their bet after each period. The two components contribute to the volatility of asset price compared to the complete markets case. In general, it depends on the correlation of the two components, that we might have asset price volatility higher or lower under incomplete versus under complete markets. However, again, given the assumption of constant aggregate endowment and I.I.D. shocks, the first component has negligible effects on asset prices. Thus the second component dominates and makes asset price volatility under incomplete markets higher than the it is under complete markets in the long run.

I measure price volatility as one-period ahead standard deviation of price. This measure is the discrete time equivalence of the continuous instant volatility, see for example Xiong and Yan (2009). Figure 7 shows the evolution of asset price volatility under the two cases, the optimists with high or low non financial wealth. The figure shows that, in the short run, asset price might be more or less volatile under complete markets than under incomplete markets, depending on the relative non-financial wealth of agents. However, in the long run, as the optimists are driven out in the complete markets equilibrium; that makes asset price volatility converge to zero. This property does not hold under in the incomplete markets equilibrium, the overly optimistic agents constantly speculate on asset price using the same asset as collateral. Asset price becomes more volatile than in the complete markets equilibrium, given that the wealth of the optimists constantly change as they win or loose their bets.

Strikingly, the smaller the non-financial wealth of the optimist is, the higher the short-run asset price volatility in the incomplete markets equilibrium but the lower the short-run asset price volatility in the complete markets equilibrium. This is because, under complete markets, it takes less time to drive out the optimists if they have lower non-financial wealth. As we increase the non-financial wealth of the optimists, we increase the short-run volatility of asset price with complete markets and decrease the short-run volatility of asset price with incomplete markets. Figure 7 plots the average asset price volatility over time for complete markets (dashed lines) and for incomplete markets (solid lines) equilibria, with different levels of "non-financial wealth" of the optimists (low and high). This figure shows that, above some certain level of non-financial wealth of the optimists, in the short-run asset price is more volatile under complete markets. But in the long run, the reverse inequality holds (right panel).

5.2.4 The financial crisis 2007-2008

Geanakoplos (2009) argues that the introduction of credit default swap (CDS) triggered the financial crisis 2007-2008. The reason is that the introduction of CDS moves the markets close to being complete. CDS allow pessimists to leverage their pessimism about the assets.

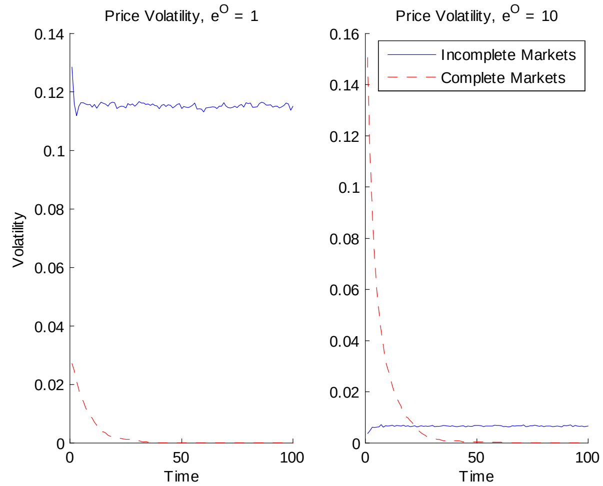


Figure 7: Asset Price Volatility Over Time

I do the same exercise here by simulating a financial markets equilibrium in its stationary state from time $t = 0$ until time $t = 50$.[36] At $t = 51$ markets suddenly become complete. In Figure 8, the left panel plots asset price level and the right panel plots asset price volatility over time. The simulation shows that asset price decreases but asset price volatility increases in the short run after the introduction of CDS. The reason for the fall in asset price is that the pessimists can now "leverage their view" with a complete set of financial assets. The reason for increasing asset price volatility in the short run is the movement in the wealth distribution toward the long-run wealth distribution, which concentrates on pessimists. Nevertheless in the long run asset price volatility goes to zero when wealth distribution fully concentrates on the pessimists.

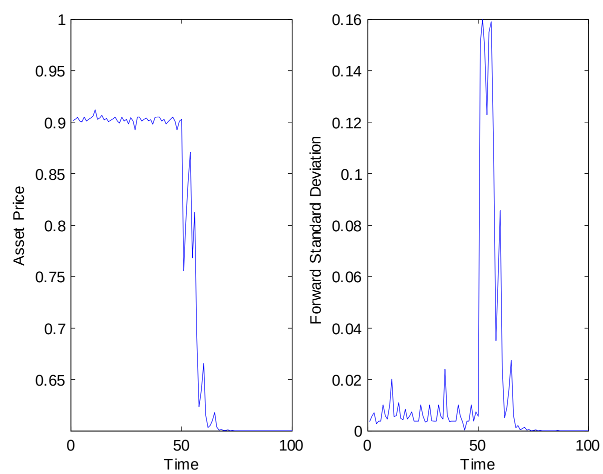


Figure 8: Financial Crisis 2007-2008

[36] In order to generate high short-run asset price volatility, I choose a high level of the optimists' endowment $e^O = 10$.

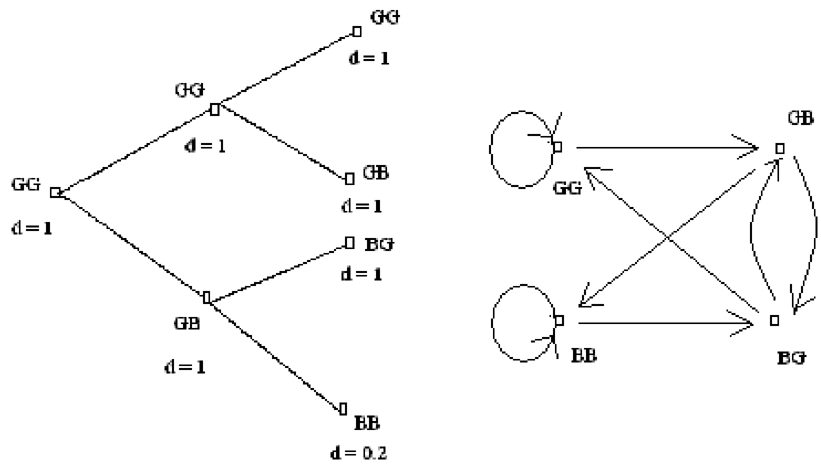


Figure 9: Evolution of the Aggregate States

5.2.5 Dynamic leverage cycles

Even though the example in Subsection 5.2.3 generates high asset price volatility, leverage is not consistent with what we observe in financial markets: high leverage in good times and low leverage in bad times, as documented in Geanakoplos (2009).

In order to generate the procyclicality of leverage, I use the insight from Geanakoplos (2009) regarding aggregate uncertainty: bad news must generate more uncertainty and more disagreement in order to reduce equilibrium leverage significantly. To formalize this type of news, I assume that after a series of good shocks, the first bad shock does not immediately reduce dividends. After this bad shock, however, dividends plunge if a second bad shock hits the economy. Therefore the first bad shock only increases uncertainty regarding dividends but not their level. In a dynamic setting, the formulation translates to a dividend process that depends not only on current exogenous shock but also on the last period exogenous shock. Therefore we need to use four exogenous shocks, instead of the two exogenous shocks in the last subsections:

$$s \in \{GG, GB, BG, BB\}.$$

Figure 9, left panel, shows that the initial bad shocks following a series of good shocks does not reduce dividends. However, dividends fall to 0.2, if a second bad shock hits the economy, i.e., the first bad shock increases uncertainty in dividends. The right panel of the figure shows the evolution through time of the exogenous states using Markov chain representation.

This uncertainty structure generates high leverage at good states GG and BG and low leverage in bad states GB and BB. Figure 10 shows this pattern of leverage. The dashed line represents leverage level in good states $s = GG$ or BG as a function of the normalized wealth distribution. The two solid lines represent leverage level in bad states $s = GB$ or BB.

In addition to the fact that uncertainty affects leverage emphasized in Geanakoplos (2009), we also learn from Figure 10 that financial wealth distribution is another impor-

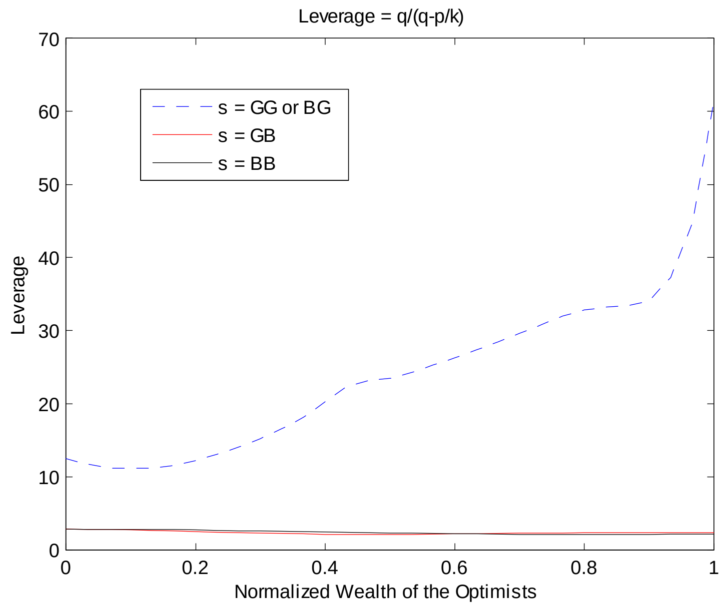


Figure 10: Leverage Cycles

tant determinant of leverage. For example, we learn from the figure that leverage decreases dramatically from good states to bad states. However, in contrast to the static version in Geanakoplos (2009), changes in the wealth distribution do not amplify the decline in leverage from good states to bad states as leverage is insensitive to the wealth distribution in bad states.

Moreover, this version of dynamic leverage cycles generates a pattern of leverage build-up in good times. Good shocks increase leverage as they increase the wealth of the optimists relative to the wealth of the pessimists and leverage is increasing the wealth of the optimists. Figure 11 shows the evolution of the wealth distribution and leverage over time. The economy starts at good state and $\omega^O = 0$. It experiences 9 consecutive good shocks from $t = 1$ to 9 and two bad shocks at $t = 10, 11$ then another 9 good shocks from $t = 12$ to 20. This figure shows that, in good states, both the wealth of the optimists and leverage increase. However their wealth and leverage plunge when bad shocks hit the economy.

5.2.6 Regulating Leverage

Subsection 5.2.3 shows that, in a incomplete markets equilibrium, when the non-financial wealth of the optimists is small relative to asset prices, variations in their wealth play an important role in driving up asset price volatility. It is then tempting to conclude that by restricting leverage, we can reduce the variation of wealth of the optimists, therefore reduce asset price volatility. However, this simple intuition is not always true by two reasons. First, restricting leverage limits the demand for asset of the optimists when their "financial wealth" is small, therefore drives down asset price. In contrast, when their "financial wealth" is large, restricting leverage does not affect the demand, thus does not affect asset price. The two channels create a potential for higher asset price volatility. Second, restricting leverage does

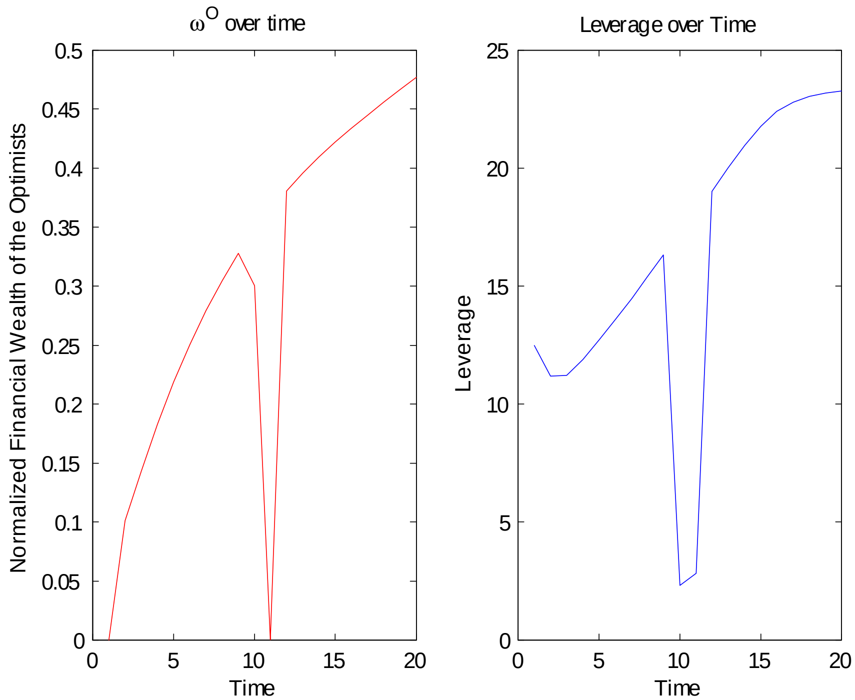


Figure 11: Leverage Cycles

reduce asset price in the short run when the optimists are poor, however in the long run they can accumulate the asset and become wealthier. High leverage requirements prevent them from falling back to the low wealth region. So in the long run, restricting leverage drives up asset price volatility due to the first reason and high long run wealth of the optimists.

To show this statement, I go to the extreme case, when leverage is strictly forbidden, i.e., there are no financial assets. Figure 12 plots the volatility of asset price as functions of the financial wealth of the optimists in two cases, with financial markets and without financial markets[37]. We can see that, with financial markets, asset volatility is higher when the optimists are poor and lower when they are rich. The reverse holds without financial markets. The numerical solution also shows that, without financial markets, the optimists always accumulate assets to move up to the high financial wealth (asset holding) region.[38] This dynamics makes asset price more volatile without financial markets then it is with financial markets.

Figure 13 shows the Monte-Carlo simulation for an economy starting in good state and $\omega^O = 0$. The figure plots the evolution of the average of the normalized financial wealth of the optimists, left panel, and asset price volatility, right panel, over time (the solid lines represent the unregulated economy and the dashed lines represent the regulated economy). As discussed above, the wealth of the optimists remains low on average in the unregulated economy but increases to a permanently high level under regulation. Thus, initially asset

[37] Without financial market, "financial wealth" is asset holding itself.

[38] With leverage, the optimists will want to hold more of the real assets using leverage, but given that they have incorrect beliefs, they will tend to lose all their shares. Notice that consumption patterns are pretty similar with or without leverage. The main difference between the two equilibria is the asset holdings.

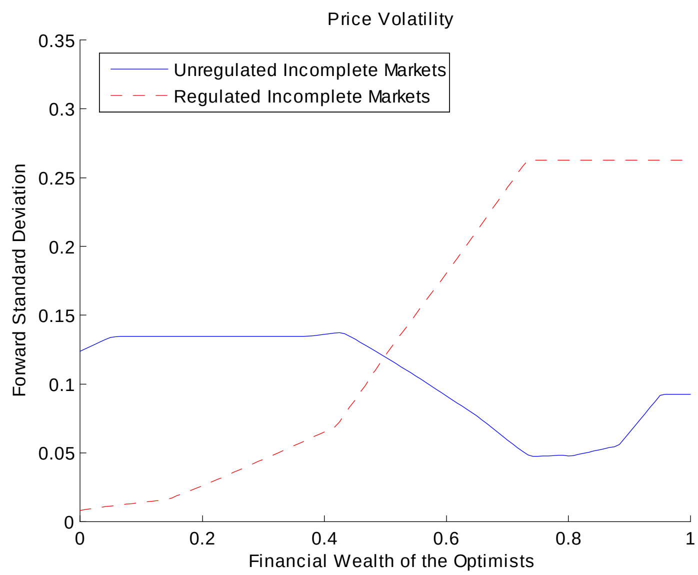


Figure 12: Asset Price Volatility in Unregulated and Regulated Economies

price volatility is higher in the unregulated economy than in the regulated economy. The reverse inequality holds, however, as over time, the wealth of the optimists increase more in the regulated economy than in the unregulated economy.

I conclude this part with two additional remarks. First, intermediate regulations can be computed using Proposition 8. If the regulator requires collateral $k \geq k_r$, then the proposition shows that in equilibrium, only the leverage level $\max(k_t^*, k_r)$ prevails. The numerical solutions for intermediate regulations confirms the conclusion in the paragraphs above. Second, regulation not only fails to reduce asset price volatility, it also reduces welfare of both types of agents as it reduces trading possibilities.

6 General model with capital accumulation

To study the effects of beliefs heterogeneity on the real economy, i.e., with aggregate capital accumulation and production, I introduce a second type of physical assets: assets in flexible supply.

6.1 Assets with fixed and flexible supplies

Adjustment cost: The set of $\mathcal{A}$ is decomposed into two subsets, $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, one with elastic supply assets, $\mathcal{A}_0$, associated with adjustment cost functions, and the other ones with fixed supply, $a \in \mathcal{A}_1$ generating divided as described in Section 2. Let A_0, A_1 respectively denote the numbers of assets with elastic and fixed supply.

We can think of assets with fixed supply, $a \in \mathcal{A}_1$, as having infinite adjustment costs, however for the rigorousness of the model, I treat them differently from the assets with elastic supply.

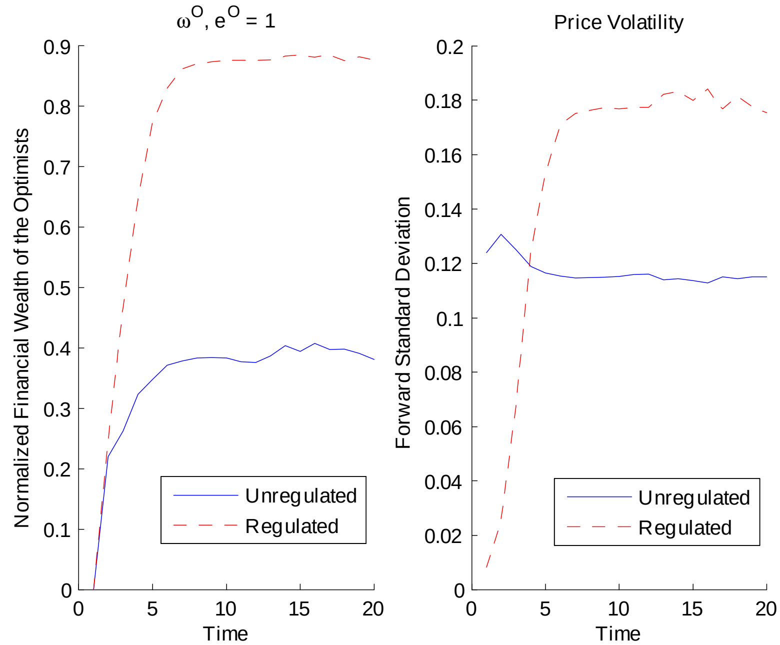


Figure 13: Wealth Distribution and Asset Price Volatility over Time

For each asset with elastic supply, $a \in \mathcal{A}_0$, in each period, k_a^n new units of asset a can be produced using k_a^o old units of asset a and $\Psi_a(k_a^n, k_a^o)$ units of the final good. The k_a^n new units are used for production in the next period. Let $q_{a,t}$ denote the ex-dividend price of each old unit of asset a, and $q_{a,t}^*$ denote the price of each new unit of asset a. Notice that $\Psi_a(k_a^n, k_a^o)$ is the final good investment associated to asset a. One example typically used in macroeconomics, representing perfectly flexible investment, is

$$\Psi_a(k_a^n, k_a^o) = k_a^n - (1-\delta_a)\, k_a^o. \tag{33}$$

Another example with non-linearity is the one used in Lorenzoni and Walentin (2009)

$$\Psi_a(k_a^n, k_a^o) = k_a^n - (1-\delta_a)\, k_a^o + \frac{\xi_a}{2}\frac{(k_a^n - k_a^o)^2}{k_a^o},$$

in which $0 < \xi_a < \min\{2(1-\delta_a), 1\}$.

We can also rewrite the adjustment cost under a more familiar form

$$k_a^n = (1-\delta_a)\, k_a^o + k_a^o \Phi\left(\frac{i_a}{k_a^o}\right), \tag{34}$$

in which i_a is real investment in terms of final good. $\Phi(.)$ is strictly increasing and weakly concave. The perfectly investment case in (33) corresponds to $\Phi(x) = x$.

I make the following standard assumption on the adjustment cost function. This assumption ensures that the profit maximization of each asset producer yields upper-hemicontinous and convex solutions.

Assumption 6 *The adjustment cost function Ψ_a is homogeneous of degree 1 and convex in $(k_a^n, -k_a^o)$. Moreover, Ψ_a is strictly increasing in k_a^n and strictly decreasing in k_a^o.*

Production: Assets with fixed supply, $a \in \mathcal{A}_1$ generate a state-dependent stream of dividend $d_a(s)$. An asset a with elastic supply can be used as an input in production with a state-dependent production function $F_a(K_a, L_a, s)$, in which K_a are units of assets of type a and L_a is labor of the type associated to the asset.

Similarly to the adjustment cost, I make the following standard assumption to ensure that the profit maximization of each final good producer yields upper-hemicontinuous and convex solutions.

Assumption 7 *The production function $F_a(K_a, L_a, s)$ is homogeneous of degree 1 and concave in (K_a, L_a) and strictly increasing in both parameters.*

One example is the standard Cobb-Douglas production function with state-dependent productivity used in the RBC literature

$$F_a(K_a, L_a, s) = A(s) K_a^{\alpha_a} L_a^{1-\alpha_a}.$$

Beside the group of consumers in Section 2, there are two other groups of agents in this economy: the asset producers and the final good producers. These producers live only for one period, therefore they do not have to make inter-temporal decisions.[39]

Asset Producers: In each state, there are A_0 representative asset producers. Asset producer $a \in \mathcal{A}_0$ produces $K_{a,t}^n$ unit of new asset from $K_{a,t-1}^o$ old units of old assets and $\Psi_a\left(K_{a,t}^n, K_{a,t-1}^o\right)$ units of final good. The producers take prices $q_{a,t}^*$ and $q_{a,t}$ as given to maximize their profit

$$\pi_t^a = \max_{\substack{K_{a,t}^n, K_{a,t-1}^o \geq 0 \\ \psi_{a,t} \geq \Psi_a\left(K_{a,t}^n, K_{a,t-1}^o\right)}} q_{a,t}^* K_{a,t}^n - \psi_{a,t} - q_{a,t} K_{a,t-1}^o. \tag{35}$$

Final Good Producers: In each state there is also A_0 representative final good producers. Producer $a \in \mathcal{A}_0$ produces $F_a(K_a, L_a, s)$ units of final good from K_a units of asset a and L_a units of labor associated to the asset[40]. The producers take rental prices $d_{a,t}$ and wages $w_{a,t}$ as given to maximize their profit

$$\pi_t^{f,a} = \max_{\substack{K_{a,t}^f, L_{a,t}, y_{a,t} \geq 0 \\ y_{a,t} \leq F_a\left(K_{a,t}^f, L_{a,t}, s_t\right)}} y_{a,t} - d_{a,t} K_{a,t}^f - w_{a,t} L_{a,t} \tag{36}$$

The consumers are still the main actors in this economy. Consumers h are endowed with a vector of labor

$$L_h = \left(L_{h,a}(s_t)\right)_{a \in \mathcal{A}_0},$$

[39] See Ljungqvist and Sargent (2004), Chapter 12, for a similar reformulation of the standard stochastic growth models, using two types of firms. Type 1 firms produce consumption good using labor and capital. Type 2 firms produce capital good using consumption good. The two types of firms corresponds to final good producers and asset producers in my model.

[40] In an alternative model, assets use the same type of labor. That model is similar to the one presented here.

$L_{h,a}$ corresponds to labor associated with asset a in flexible supply.

The consumer maximizes her intertemporal expected utility with the per period utility function $U_h(.,.) : \mathbf{R}^+ \times (\mathbf{R}^+)^{A_0} \longrightarrow \mathbf{R}$ satisfies

Assumption 8 *U_h is concave and strictly increasing in $(c, -l)$.*

Consumer h takes sequences of prices as given and solves

$$\max E_0^h \left[\sum_{t=0}^{\infty} \beta_h^t U_h \left(c_t^h, l_{a.t}^h \right) \right]$$

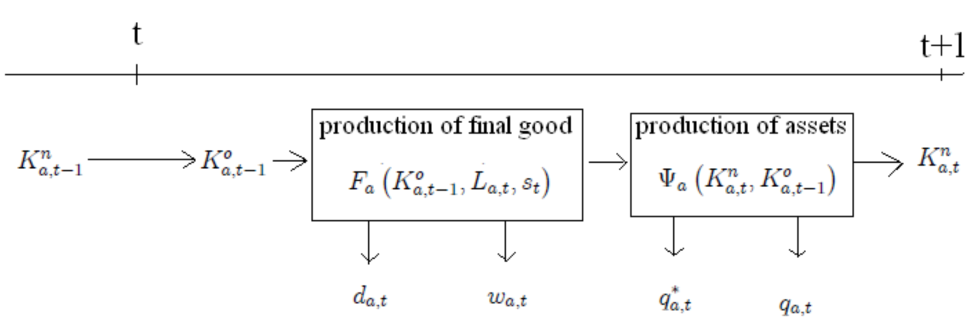
and in each history s^t, she is subject to the budget constraint

$$\begin{aligned} & c_t^h + \sum_{a \in \mathcal{A}_1} q_{a,t} k_{a,t}^h + \sum_{a \in \mathcal{A}_0} q_{a,t}^* k_{a,t}^h + \sum_{j=1}^{J} p_{j,t} \phi_{j,t}^h \\ \leq \; & e_t^h + \sum_{a \in \mathcal{A}_0} w_{a,t} l_{a,t}^h + \sum_{j=1}^{J} f_{j,t} \phi_{j,t-1}^h + \\ & + \sum_{a \in \mathcal{A}_0} \left(q_{a,t} + d_{a,t} \right) k_{a,t-1}^h + \sum_{a \in \mathcal{A}_0} \Pi_a^h + \sum_{a \in \mathcal{A}_0} \Pi_a^{f,h} \\ & + \sum_{a \in \mathcal{A}_1} \left(q_{a,t} + d_{a,t} \right) k_{a,t-1}^h, \end{aligned} \tag{37}$$

and the collateral constraints (5).

In the budget constraint (37), consumer h can trade old units of real asset a at price $q_{a,t}$, buy new units of asset $k_{a,t}^h$ for time $t+1$ at price $q_{a,t}^*$. Finally, she works at the wage $w_{a,t}$ in each production sector a. She also receives her shares of profit from the asset producer and final good producer at time t, Π_a^h and $\Pi_a^{f,h}$. However, given the homogeneity of the production functions, these profits should be zero in equilibrium.

Within a period, timing of decisions and actions taken by the agents are summarized in the following figure:



A number of features are worth noting in this setup: The demand of the consumers for new assets is similar to Tobin's Q theory of investment. They weigh the perceived marginal benefit of one additional unit of an asset a: future rental price, $d_{a,t+1}$, and future resale value $q_{a,t+1}$, against the marginal cost of buying one new unit of that asset at price $q^*_{a,t}$. The total demand for new units of asset a from the consumers is decreasing in price $q^*_{a,t}$ and the supply of the asset from the asset producers is increasing in $q^*_{a,t}$. In equilibrium both $q^*_{a,t}$ and $K^n_{a,t}$ are determined simultaneously. For instance, if the consumers expect low future resale price of an asset, they will demand less for new units of the asset. This low demand leads to low current price and low investment in the asset.

In this environment, I define an equilibrium as follows

Definition 4 *An* ***incomplete markets equilibrium*** *for an economy with initial asset holdings*

$$\left\{k^h_{a,0}\right\}_{h\in\{1,2,\dots,H\}}$$

and initial shock s_0 is a collection

$$\begin{aligned}
&(\left\{c^h_t\left(s^t\right), l^h_{a,t}\left(s^t\right), k^h_{a,t}\left(s^t\right), \phi^h_{j,t}\left(s^t\right)\right\}_{h\in\{1,2,\dots,H\}} \\
&\left\{K^n_{a,t}\left(s^t\right), K^o_{a,t}\left(s^t\right), \psi_{a,t}\left(s^t\right)\right\}_{a\in\mathcal{A}_0} \\
&\left\{K^f_{a,t}\left(s^t\right), L_{a,t}\left(s^t\right), y_{a,t}\left(s^t\right)\right\}_{a\in\mathcal{A}_0} \\
&\left\{q^*_{a,t}\left(s^t\right), q_{a,t}\left(s^t\right), d_{a,t}\left(s^t\right), w_{a,t}\left(s^t\right)\right\}_{a\in\mathcal{A}_0} \\
&\left\{q_{a,t}\left(s^t\right)\right\}_{a\in\mathcal{A}_1}, \left\{p_{j,t}\left(s^t\right)\right\}_{j\in J_t(s^t)})
\end{aligned}$$

satisfying the following conditions
i) Asset markets and labor market for each asset with elastic supply $a\in\mathcal{A}_0$ in each period clear:
Demand by the consumers for new units of assets a equals supply of new units by the asset a producer:

$$\sum_{h=1}^{H} k^h_{a,t}\left(s^t\right) = K^n_{a,t}\left(s^t\right),$$

Demand by the asset a producer for old units of assets a equal supply of old units by the consumers:

$$K^o_{a,t}\left(s^t\right) = \sum_{h=1}^{H} k^h_{a,t-1}\left(s^t\right),$$

Demand by the asset a final good producer for old units of assets a equal supply of old units by the consumers:

$$K^f_{a,t}\left(s^t\right) = \sum_{h=1}^{H} k^h_{a,t-1}\left(s^t\right).$$

Labor demand by the asset a final good producer equal total labor supply by the consumers:

$$L_{a,t} = \sum_{h=1}^{H} L^h_{a,t}\left(s^t\right).$$

Market for each real asset in fixed supplied $a \in \mathcal{A}_1$ clears

$$K_a = \sum_{h=1}^{H} k_{a,t}^h \left(s^t\right).$$

Market for each financial asset j clears:

$$\sum_{h=1}^{H} \phi_{j,t}^h \left(s^t\right) = 0.$$

ii) For each consumer h, the allocation $\left\{c_t^h\left(s^t\right), l_a^h\left(s^t\right), k_{a,t}^h\left(s^t\right), \phi_{j,t}^h\left(s^t\right)\right\}$ solves the individual maximization problem subject to the budget constraint, (37), and the collateral constraint, (5). Asset producers and final good producers maximize their profit as in (35) and (36).

The case in which there is only one asset with perfect elastic supply, i.e., adjustment cost described in (33) and no financial assets corresponds to Krusell and Smith (1998)'s model as we allow for both aggregate shocks and incorporate idiosyncratic shocks to each individual in each exogenous shock $s \in \mathcal{S}$, and allow for a large number of agents. Therefore we can apply the existence proof in section 3 to show the existence of the recursive equilibrium in their original paper.

As a benchmark I also study equilibria under complete financial markets. Consumers can borrow and lend freely by buying and selling Arrow-Debreu state contingent securities, only subject to the no-Ponzi condition. In each node s^t, there are S financial securities. Financial security s deliver one unit of final good if state s happens at time $t+1$ and zero otherwise. Let $p_{s,t}$ denote time t price and let $\phi_{s,t}^h\left(s^t\right)$ denote consumer h's holding of this security. The budget constraint (37) of consumer h becomes

$$\begin{aligned} & c_t^h + \sum_{a \in \mathcal{A}_1} q_{a,t} k_{a,t}^h + \sum_{a \in \mathcal{A}_0} q_{a,t}^* k_{a,t}^h + \sum_{s \in \mathcal{S}} p_{s,t} \phi_{s,t}^h \\ \leq \ & e_t^h + \sum_{a \in \mathcal{A}_0} w_{a,t} l_{a,t}^h + \phi_{s_t,t-1}^h + \\ & + \sum_{a \in \mathcal{A}_0} \left(q_{a,t} + d_{a,t}\right) k_{a,t-1}^h + \sum_{a \in \mathcal{A}_0} \Pi_a^h + \sum_{a \in \mathcal{A}_0} \Pi_a^{f,h} \\ & + \sum_{a \in \mathcal{A}_1} \left(q_{a,t} + d_{a,t}\right) k_{a,t-1}^h \end{aligned} \tag{38}$$

Definition 5 *A* ***complete markets equilibrium*** *is defined similarly to incomplete markets equilibrium except that each consumer solves her individual maximization problem subject to the budget constraint (38) and the no-Ponzi condition, instead of the collateral constraint (5).*

6.2 Markov Equilibrium

6.2.1 The state space

I define the financial wealth of each agent by

$$\omega_t^h = \frac{\sum_a (q_{a,t} + d_{a,t}) k_{a,t-1}^h + \sum_j \phi_{j,t}^h f_{j,t-1}}{\sum_a (q_{a,t} + d_{a,t}) K_{a,t-1}^o}.$$

Let $\omega(s^t) = (\omega^1(s^t), ..., \omega^H(s^t))$.

As in Section 3, in equilibrium $\omega(s^t)$ always lies in the (H-1)-dimensional simplex Ω, i.e., $\omega^h \geq 0$ and $\sum_{h=1}^H \omega^h = 1$. I will show that, under conditions detailed in Subsubsection 6.2.3 below, there exists a Markov equilibrium over a compact state space. I look for an equilibrium in which equilibrium prices and allocations depend only on the states $(s_t, \omega_t, K_{t-1}^o) \in \mathcal{S} \times \Omega \times E$, in which

$$E = \prod_{a \in \mathcal{A}_0} [0, \overline{K}_a].$$

$K_a^o \in [0, \overline{K}_a]$ are the total old units of assets with elastic supply at the beginning of a period.

Let the state space $\mathcal{X}$ consist of all exogenous and endogenous variables that occur in the economy at some node σ, i.e., $\mathcal{X} = \mathcal{S} \times \mathcal{V}$, where $\mathcal{S}$ is the finite set of exogenous shocks and $\mathcal{V}$ is the set of all possible endogenous variables.

In each node σ, an element $v(\sigma) \in \mathcal{V}$ includes: the normalized wealth distribution $(\omega_h(\sigma))_{h \in \mathcal{H}} \in \Omega$, the total old units of assets with elastic supply $(K_a^o)_{a \in \mathcal{A}_0} \in E$; together with consumers' decisions: consumption, $H + HA_0$ current consumption and labor supply $(c^h(\sigma), l_a^h(\sigma))_{h \in \mathcal{H}}$, $HA + HJ$ real and financial asset holdings $(k_a^h(\sigma), \phi_j^h(\sigma))_{h \in \mathcal{H}}$. It also includes the $4A_0$ current prices of new units of elastic supply assets, the prices of old units of these assets, the rental prices and wages associated with these assets

$$(q_a^*(\sigma), q_a(\sigma), w_a(\sigma), d_a(\sigma))_{a \in \mathcal{A}_0},$$

and A_1 prices of assets with fixed supply $(q_a(\sigma))_{a \in \mathcal{A}_1}$. Finally it includes J prices of the financial assets $(p_j(\sigma))_{j \in J}$. Therefore $\mathcal{V} = \Omega \times E \times \widehat{\mathcal{V}}$ with

$$\widehat{\mathcal{V}} = \mathbf{R}_+^H \times \mathbf{R}_+^{HA_0} \times \mathbf{R}_+^{AH} \times \mathbf{R}^{JH} \times \mathbf{R}_+^{4A_0} \times \mathbf{R}_+^{A_1} \times \mathbf{R}_+^J \tag{39}$$

the set of endogenous variables other than the wealth distribution and total old quantities of assets with elastic supply.

Finally, let $\mathcal{X} \subset \mathcal{V}$ denote the set of vectors of all the endogenous variables that satisfies: 1) financial and asset markets clears, 2) producers maximize their profit and 3) the budget constraints of consumers bind. Formally,

$$\begin{aligned} \sum_h \phi_j^h &= 0 \\ \sum k_a^h &= K_a \ \forall a \in \mathcal{A}_1 \end{aligned}$$

and

$$l_a^h = L_{h,a}.$$

In addition, for each $a \in \mathcal{A}_0$, given $K_a^n = \sum_h k_a^h$ and $L_a = \sum_h l_a^h$ we have

$$\left(K_a^n, K_a^o, \psi_a\right) \in \underset{\substack{\widetilde{K}_a^n, \widetilde{K}_a^o \geq 0 \\ \widetilde{\psi}_a \geq \Psi_a\left(\widetilde{K}_a^n, \widetilde{K}_a^o\right)}}{\arg\max} \quad q_a^* \widetilde{K}_a^n - \widetilde{\psi}_a - q_a \widetilde{K}_a^o \tag{40}$$

and

$$\left(K_a^o, L_a, y_a\right) \in \underset{\substack{\widetilde{K}_a^f, \widetilde{L}_a, \widetilde{y}_a \geq 0 \\ \widetilde{y}_a \leq F_a\left(\widetilde{K}_a^f, \widetilde{L}_a, s\right)}}{\arg\max} \quad \widetilde{y}_a - d_a \widetilde{K}_a^f - w_a \widetilde{y}_a \tag{41}$$

and consumers' budget constraints hold with equality[41][42]

$$c^h = e^h + w \cdot l + \omega^h \left(q + d\right) \cdot K^o - q^* \cdot k - p \cdot \phi. \tag{42}$$

Notice that profit maximizations (40),(41) and binding budget constraints imply that the good market clears

$$\sum_h c^h + \sum_{a \in \mathcal{A}_0} \Psi_a\left(K_a, K_a^o\right) = \sum_h e^h + \sum_{a \in \mathcal{A}_0} F_a\left(K_a^o, L_a, s\right).$$

6.2.2 Markov Equilibrium Definition

In order to define a Markov equilibrium, I use the same definition of expectation correspondence as in 3 except

$$\omega_s^{h+} = \frac{k^h \cdot \left(q_s^+ + d_s^+\right) + \sum_{j \in \mathcal{J}} \phi_j^h \min\left\{b_j\left(s\right), \sum_{a \in A} k_a^j \left(q^+ + d^+\right)\right\}}{\sum_a \left(q^+ + d^+\right) \cdot K^+}.$$

Definition 6 *A Markov equilibrium consists of a (non-empty valued) policy correspondence, P, and a transition function F*

$$P : \mathcal{S} \times \Omega \times E \rightrightarrows \widehat{\mathcal{V}}$$

and

$$F : graph\left(P\right) \longrightarrow \mathcal{V}^{\mathcal{S}}$$

such that $graph\left(P\right) \subset \mathcal{X}$ and for all $x \in graph\left(P\right)$ and all $s \in \mathcal{S}$ we have $F\left(x\right) \subset g\left(x\right)$ and $\left(s, F_s\left(x\right)\right) \in graph\left(P\right)$.

Lemma 4 *A Markov equilibrium is an incomplete markets equilibrium according to Definition 4.*

Proof. This result is similar to the one in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994). We only need to show that the first order conditions as represented by Lagrange multipliers are sufficient to ensure the optimal solution of the consumers. This holds because the optimization each consumer faces is a convex maximization problem. ■

[41] With some abuse of notation, we use $q_a^* = q_a$ for $a \in \mathcal{A}_1$.

[42] Profit maximization conditions (40) and (41) imply zero profits from the producers, hence the absence of these profits in the consumers' budget constraint.

6.2.3 Existence and Property of Markov Equilibrium

I need the following assumptions, in addition to Assumption 5 in Subsection 3.3, evaluated at the upper bound of the total supply of flexible supply assets defined below.

Assumption 9 *There exists a $\overline{K}_a > 0$ for each $a \in \mathcal{A}_0$ such that*

$$\Psi_a\left(\overline{K}_a, \overline{K}_a\right) \geq \max_{s \in \mathcal{S}} \overline{e}_a(s), \tag{43}$$

where

$$\begin{aligned} \overline{e}_a(s) &= \sum_{h=1}^{H} e^h(s) + \sum_{a' \in \mathcal{A}_1} d_{a'}(s) K_{a,0} + \sum_{a' \in \mathcal{A}_0} F_{a'}\left(\overline{K}_{a'}, \sum_{h=1}^{H} L^h_{a'}, s\right) \\ &\quad + \sum_{a' \in \mathcal{A}_0 \setminus \{a\}} \Psi_{a'}\left(0, \overline{K}_{a'}\right) \end{aligned}$$

Assumption 10 *The first-derivatives of Ψ_a are bounded over $\left[0, \overline{K}_a\right]^2$.*

The first assumption ensures that total quantities of elastic-supply assets are bounded. For example, when we have only one elastic-supply asset and its supply is perfectly elastic, i.e., the adjustment cost function is given by the flexible investment function (33) and the associated production is Cobb-Douglas with $\alpha_a \in (0,1)$. Then inequality (43) is equivalent to

$$\delta_a \overline{K}_a > const + A\left(\overline{K}_a\right)^{\alpha_a} L_a^{1-\alpha_a}$$

which must be true for $\overline{K}_a$ large enough. This is also the way one obtains an upper bound for capital in a neoclassical growth model. The second assumption ensures that prices of new and old assets are bounded in equilibrium as they correspond to the first-derivatives of Ψ_a. For example, (33) gives

$$\begin{aligned} \frac{\partial \Psi_a\left(K_a^n, K_a^o\right)}{\partial K^n} &= 1 \\ \frac{\partial \Psi_a\left(K_a^n, K_a^o\right)}{\partial K^o} &= -\left(1 - \delta_a\right). \end{aligned}$$

The details of the existence proof are in Appendix B.

Proposition 9 *When aggregate endowment and aggregate productivity are constant, and shocks are I.I.D., long run investment volatility is higher under incomplete markets than it is under complete markets.*

Proof. In the long run, under complete markets, the economy converges to the one with homogenous beliefs because agents with incorrect beliefs will eventually be driven out of the markets. We can thus find a representative agent. Standard arguments for representative agent economy imply that asset prices are constant and levels of investment converge to their steady state levels. When we have assets in elastic supply, but with constant productivity, as in the neoclassical growth model, the total quantity of an asset a in fixed supply should converge to the steady-state level K_a^* which is determined by the

$$\frac{\partial \Psi_a(K_a^*, K_a^*)}{\partial K_a^n} = \beta \left(-\frac{\partial \Psi_a(K_a^*, K_a^*)}{\partial K_a^o} + F_{a,K}(K_a^*, L_a) \right)$$

and therefore the investment associated to this asset converges to $I_a^* = \Psi_a(K_a^*, K_a^*)$. Hence, under complete markets, asset price volatility and investment volatility converge to zero in the long run. Under incomplete markets, asset price volatility and investment volatility remain well above zero as aggregate shocks constantly change the wealth distribution, which, in turn, changes asset prices and investment. ■

6.2.4 Relationship to recursive equilibria

When we do not have financial assets and there is only one real asset, then Markov equilibria are recursive equilibria. This is also true when initially agents hold the same fraction of each assets. However, in general, Markov equilibria are not recursive equilibria. But in Kubler and Schmedders (2003), subsection 4.4 shows that we can construct recursive equilibria from Markov equilibria if we can extract a continuous mapping from the policy correspondence.

As an important special case discussed after Definition 1 of incomplete markets equilibrium, the economy in Krusell and Smith (1998) corresponds to the economy here with one asset in perfectly elastic supply and without financial markets. The existence of a Markov equilibrium implies the existence of a recursive equilibrium. Indeed, given that there are no financial markets and only one asset. The "normalized financial wealth" ω_t^h becomes $\frac{k_{a,t-1}^h}{K_{a,t-1}}$ the fraction of capital asset holding. Together with the total quantity of capital, $K_{a,t}$, the state variables $(s_t, \omega_t, K_{a,t-1})$ is equivalent to $\left(s_t, \left(k_{a,t-1}^h\right)\right)$, the aggregate state and capital holdings of each agent in the definition of recursive equilibria, in page 874 of Krusell and Smith (1998). In a recent paper, Miao (2006) shows the existence of recursive equilibrium however he has to include future expected discounted utilities of agents in the state-space. In addition, he wrote in page 291, that the question whether a recursive equilibrium in Krusell and Smith (1998) exists remains an open question. The existence proof here suggests a positive answer to that question under the form of "generalized recursive equilibrium". A "generalized recursive equilibrium" might not be a simple recursive equilibrium because the mapping might be multi-valued.

7 Conclusion

In this paper I develop a dynamic general equilibrium model to examine the effects of belief heterogeneity on the survival of agents and on asset price and investment volatility under different financial markets structures. I show that, when financial markets are endogenously incomplete, agents with incorrect beliefs survive in the long run. The survival of these agents leads to higher asset price and investment volatility. This result contrasts with the frictionless complete markets case, in which agents holding incorrect beliefs are eventually driven out and as a result, asset prices and investment exhibit lower volatility.

In addition, I show the existence of stationary Markov equilibria in this framework with incomplete financial markets and with general production and capital accumulation technology. I also develop an algorithm for computing the equilibria. As a result, the framework can be readily used to investigate questions about the interaction between financial markets and the macroeconomy. For instance, it would be interesting in future work to apply these methods in calibration exercises using more rigorous quantitative asset pricing techniques, such as in Alvarez and Jermann (2001). This could be done by allowing for uncertainty in the growth rate of dividends rather than uncertainty in the levels, as modeled in this paper, in order to match the rate of return on stock markets and the growth rate of aggregate consumption. Such a model would provide a set of moment conditions that could be used to estimate relevant parameters using GMM as in Chien and Lustig (2009). A challenge in such work, however, is that finding the Markov equilibria is computationally demanding. I follow that path in Cao, Chen, and Scott (2011).

A second avenue for further research is to examine more normative questions in the framework developed in this paper. My results suggest, for example, that financial regulation aimed at reducing asset price and real investment volatility should be state-depedent, as conjectured by Geanakoplos (2009). It would also be interesting to consider the effects of other intervention policies, such as bail-out or monetary policies.

8 Appendices

Appendix A: Analysis of survival and disappearance under complete markets.

Proof of Proposition 1. This is an application of Proposition 2 in Sandroni (2000). We need to show that, almost surely, agents h with $\pi^h = \pi$ eventually make accurate predictions, i.e.,

$$\lim_{t \longrightarrow \infty} \frac{P_h\left(s^t\right)}{P\left(s^t\right)} > 0$$

and agents with $\pi^h \neq \pi$ do not, i.e.,

$$\lim_{t \longrightarrow \infty} \frac{P_h\left(s^t\right)}{P\left(s^t\right)} = 0.$$

The fact that P a.s. $\lim_{t \longrightarrow \infty} \frac{P_h\left(s^t\right)}{P\left(s^t\right)}$ for any h exists follow from Lemma 2 in his paper, using the Martingale Convergence Theorem. Now in order to prove that agents with $\pi^h \neq \pi$ will not make accurate predictions. We show that, almost surely, there exists a sequence s^t $\left\|P_{s^t}\left(A\right) - P^h_{s^t}\left(A\right)\right\| > \epsilon$. Let $\pi^h\left(s, s'\right) \neq \pi\left(s, s'\right)$ then infinitely often s_t will reach s as S is an ergodic set. For s^t such that $s_t = s$ let A be the set $s_{t+1} = s$

$$\left\|P_{s^t}\left(A\right) - P^h_{s^t}\left(A\right)\right\| = \left|\pi^h\left(s, s'\right) - \pi^h\left(s, s'\right)\right| > \epsilon > 0 \tag{44}$$

Notice that we cannot directly apply Propositions 3,4, and 5 in Sandroni (2000) as they require (44) holds for all t. But at some t when $\pi\left(s_t, .\right) = \pi^h\left(s_t, .\right)$ the two probabilities are actually the same so (44) does not hold. This case is prevalent when we have time varying beliefs dispersion, in which people share common beliefs in normal times. ■

Proof of Propostion 4. The asset price is the presented discounted value of dividends weighted by the stochastic discount factor:

$$q_a\left(s^t\right) = \sum_{r=0}^{\infty} P_h\left(s^{t+r}|s^t\right) \frac{U'_h\left(c_h\left(s^{t+r}\right)\right)}{U'_h\left(c_h\left(s^t\right)\right)} d_a\left(s_{t+r}\right).$$

We know that

$$\frac{U'_h\left(c_h\left(s^{t+r}\right)\right)}{U'_h\left(c_h\left(s^0\right)\right)} = \frac{U'_{h'}\left(c_{h'}\left(s^{t+r}\right)\right)}{U'_{h'}\left(c_{h'}\left(s^0\right)\right)}$$

or

$$\frac{U'_h\left(c_h\left(s^{t+r}\right)\right)}{U'_{h'}\left(c_{h'}\left(s^{t+r}\right)\right)} = \frac{U'_h\left(c_h\left(s^0\right)\right)}{U'_{h'}\left(c_{h'}\left(s^0\right)\right)}$$

so

$$c_h\left(s^{t+r}\right) = C_h\left(C_I\left(s^{t+r}\right)\right)$$

for

$$h \in I$$

and

$$C_I\left(s^{t+r}\right) = \sum_{h \in I} c_h\left(s^{t+r}\right).$$

Applying Proposition 1, we have, almost surely,

$$\lim_{t \longrightarrow \infty} \left|C_I\left(s^t\right) - e\left(s_t\right)\right| = 0$$

or

$$\lim_{t\longrightarrow\infty}\left|c_h\left(s^t\right)-C_h\left(e\left(s_t\right)\right)\right|=0$$

so

$$\lim_{t\longrightarrow\infty}\left|q_a\left(s^t\right)-\overline{q}_a\left(s_t\right)\right|=0$$

where

$$\overline{q}_a\left(s_t\right)=\sum_{r=0}^{\infty}P\left(s^{t+r}|s^t\right)\frac{U_h'\left(C_h\left(e\left(s_{t+r}\right)\right)\right)}{U_h'\left(C_h\left(e\left(s_t\right)\right)\right)}d_a\left(s_{t+r}\right).$$

This is asset price in an complete markets economy.[43] ■

Appendix B: Existence proof

To prove the existence of equilibrium in finite horizon, I allow utility to be dependent of labor decision. So per period utility of agent h is $U_h\left(c,L_h-l\right):\left(\mathbf{R}^+\right)^2\longrightarrow\mathbf{R}$ over consumption and leisure. I replace Assumption 3 by the following Assumption

Assumption 3b: $U_h\left(c,l\right)$ is strictly increasing in c, non-decreasing in l and concave in $\left(c,l\right)$.

Definition 7 *An allocation*

$$\left(\begin{array}{c} c_t^h\left(s^t\right),k_{a,t}^h\left(s^t\right),l_{a,t}^h\left(s^t\right),\phi_{j,t}^h\left(s^t\right)\\ K_{a,t}^n\left(s^t\right),K_{a,t}^o\left(s^t\right),\psi_{a,t}\left(s^t\right)\\ K_{a,t}^f\left(s^t\right),L_{a,t}\left(s^t\right),y_{a,t}\left(s^t\right)\end{array}\right)$$

together with the no default penalty defined in (1), is attainable if consumptions, real asset holdings, labor decision from the consumers, new and old real assets decision from the real asset producers and capital and labor decisions of final good producers are positive. The resources constrained are satisfied

$$L_a^h\geq l_{a,t}^h\left(s^t\right)$$
$$\psi_{a,t}\left(s^t\right)\geq\Psi_a\left(K_{a,t}^n\left(s^t\right),K_{a,t}^o\left(s^t\right)\right)$$
$$F\left(K_{a,t}^f\left(s^t\right),L_{a,t}\left(s^t\right),s\right)\geq y_{a,t}\left(s^t\right)$$

($\psi_{a,T}\left(s^T\right)\geq\Psi_a\left(0,K_{a,T}^o\left(s^T\right)\right)$ *given* $K_{a,T+1}\left(s^T\right)=0$*) and excess demands are negative:*
First, excess demands on the good markets are negative:

$$\sum_{h=1}^{H}c_t^h+\sum_{a\in\mathcal{A}_0}\psi_{a,t}-\sum_{h=1}^{H}e_t^h-\sum_{a\in\mathcal{A}_1}d_{a,t}\sum_{h}k_{a,t-1}^h-\sum_{a\in\mathcal{A}_0}y_{a,t}-\sum_{\substack{j=1\\ f_{j,t}=b_{j,t}}}^{J}\left(\sum_{h=1}^{H}\phi_{j,t-1}^h\right)b_{j,t}\leq 0$$

Second, for $a\in\mathcal{A}_0$

$$\sum_{h=1}^{H}k_{a,t}^h-K_{a,t}^n\leq 0$$

$$K_{a,t}^o-\sum_{h=1}^{H}k_{a,t-1}^h-\sum_{\substack{j=1\\ f_{j,t}<b_{j,t}}}^{J}\left(\sum_{h=1}^{H}\phi_{j,t-1}^h\right)k_{j,t-1}^a\leq 0$$

[43] See Ljungqvist and Sargent (2004) for a standard treatment of asset pricing under complete markets.

$$K^f_{a,t} - \sum_{h=1}^{H} k^h_{a,t-1} - \sum_{\substack{j=1\\ f_{j,t}<b_{j,t}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t-1}\right) k^a_{j,t-1} \leq 0$$

$$L_{a,t} - \sum_{h=1}^{H} l^h_{a,t} \leq 0$$

$$\sum_{h=1}^{H} \phi^h_{j,t} \leq 0. \tag{45}$$

for $a \in \mathcal{A}_1$

$$\sum_{h=1}^{H} k^h_{a,0} - K_{a,-1} \leq 0.$$

in each time-state t, s^t with $0 < t < T$. For the initial period there is no explicit initial debt and the aggregate supply of asset a is $K_{a,-1}$ so

$$\sum_{h=1}^{H} c^h_0 + \sum_{a\in\mathcal{A}_0} \psi_{a,0} - \sum_{h=1}^{H} e^h_0 - \sum_{a\in\mathcal{A}_1} d_{a,0}K_{a,-1} - \sum_{a\in\mathcal{A}_0} y_{a,0} \leq 0$$

For $a \in \mathcal{A}_1$

$$\sum_{h=1}^{H} k^h_{a,0} - K_{a,-1} \leq 0$$

$$K^o_{a,0} - K_{a,-1} \leq 0$$

$$K^f_{a,0} - K_{a,-1} \leq 0$$

$$L_{a,0} - \sum_{h=1}^{H} l^h_{a,0} \leq 0$$

$$\sum_{h=1}^{H} \phi^h_{j,0} \leq 0. \tag{46}$$

For $a \in \mathcal{A}_0$

$$\sum_{h=1}^{H} k^h_{a,0} - K^n_{a,0} \leq 0$$

For $t = T$, there is no financial assets that pay-off at $T+1$, so

$$\sum_{h=1}^{H} c^h_T + \sum_{a\in\mathcal{A}_0} \psi_{a,T} - \sum_{h=1}^{H} e^h_T - \sum_{a\in\mathcal{A}_1} d_{a,T} \sum_{h} k^h_{a,t-1} - \sum_{a\in\mathcal{A}_0} y_{a,T} - \sum_{\substack{j=1\\ f_{j,T}=b_{j,T}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,T-1}\right) b_{j,T} \leq 0$$

For $a \in \mathcal{A}_1$

$$K^f_{a,T} - \sum_{h=1}^{H} k^h_{a,T-1} - \sum_{\substack{j=1\\ f_{j,T}<b_{j,T}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,T-1}\right) k^a_{j,T-1} \leq 0$$

$$L_{a,T} - \sum_{h=1}^{H} l_{a,T}^{h} \leq 0. \tag{47}$$

For $a \in \mathcal{A}_0$

$$K_{a,T}^{o} - \sum_{h=1}^{H} k_{a,T-1}^{h} - \sum_{\substack{j=1 \\ f_{j,T} < b_{j,T}}}^{J} \left(\sum_{h=1}^{H} \phi_{j,T-1}^{h} \right) k_{j,T-1}^{a} \leq 0$$

Lemma 5 *The set of attainable allocations is bounded.*

Proof. We prove this Lemma by induction in t.
Before all, notice that given

$$\sum_{h} k_{a,t}^{h} - \sum_{h} k_{a,t-1}^{h} - \sum_{f_{j,t} < b_{j,t}} k_{a,t}^{j} \sum_{h} \phi_{j,t-1}^{h} \leq 0$$

for each $a \in \mathcal{A}_1$, and $t \leq T$, we have

$$\begin{aligned} \sum_{h} k_{a,t}^{h} &\leq \sum_{h} k_{a,t-1}^{h} + \sum_{f_{j,t} < b_{j,t}} k_{a,t}^{j} \sum_{h} \phi_{j,t-1}^{h} \\ &\leq \sum_{h} k_{a,t-1}^{h} \\ &\leq .. \\ &\leq \sum_{h} k_{a,0}^{h} \leq K_{a,-1} \end{aligned}$$

Step 1 $t \longmapsto t+1$: Suppose there is an M_t such that for each attainable allocation associates with an economy that satisfies

$$\begin{gathered} M_t \geq c_t^h \left(s^t\right) \geq 0 \\ M_t \geq k_{a,t-1}^{h} \left(s^t\right) \geq 0 \\ M_t \geq K_{a,t}^{n} \left(s^t\right) \geq 0 \\ M_t \geq K_{a,t}^{o} \left(s^t\right) \geq 0 \\ M_t \geq K_{a,t+1}^{f} \left(s^{t+1}\right) \geq 0 \\ M_t \geq y_{a,t+1} \left(s^{t+1}\right) \geq 0 \\ M_t \geq \left|\psi_{a,t} \left(s^t\right)\right| \end{gathered}$$

we show that the statement holds at $t+1 \leq T$ by using the system of inequalities (45) and (47):
For $a \in \mathcal{A}_0$ we have

$$K_{a,t+1}^{o} - \sum_{h=1}^{H} k_{a,t}^{h} - \sum_{\substack{j=1 \\ f_{j,t+1} < b_{j,t+1}}}^{J} \left(\sum_{h=1}^{H} \phi_{j,t}^{h} \right) k_{j,t}^{a} \leq 0$$

and

$$\sum_{h=1}^{H} \phi_{j,t}^{h} \leq 0,$$

therefore

$$K^o_{a,t+1} \leq HM_t = M^o_{t+1}.$$

Similarly

$$K^f_{a,t+1} - \sum_{h=1}^{H} k^h_{a,t} - \sum_{\substack{j=1 \\ f_{j,t+1} < b_{j,t+1}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t} \right) k^a_{j,t} \leq 0$$

therefore

$$K^f_{a,t+1} \leq M^o_{t+1}.$$

Besides,

$$\begin{aligned} \psi_{a,t+1} &\geq \Psi_a \left(K^n_{a,t+2}, K^o_{a,t+1} \right) \\ &\geq \Psi_a \left(0, M^o_{t+1} \right) = -M^{\psi-}_{a,t+1}. \end{aligned}$$

Second

$$\sum_{h=1}^{H} c^h_{t+1} + \sum_{a \in \mathcal{A}_0} \psi_{a,t+1} - \sum_{h=1}^{H} e^h_{t+1} - \sum_{a \in \mathcal{A}_1} d_{a,t+1} \sum_{h} k^h_{a,t-1} - \sum_{a \in \mathcal{A}_0} y_{a,t+1} - \sum_{\substack{j=1 \\ f_{j,t+1} = b_{j,t+1}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t} \right) b_{j,t+1} \leq 0$$

and

$$\sum_{h=1}^{H} \phi^h_{j,t} \leq 0$$

implies

$$\sum_{h=1}^{H} c^h_{t+1} + \sum_{a \in \mathcal{A}_0} \psi_{a,t+1} \leq \sum_{h=1}^{H} e^h_{t+1} + \sum_{a \in \mathcal{A}_1} d_{a,t+1} K_{a,-1} + A_0 M_t.$$

Given that $c^h_{t+1} \geq 0$, for $a \in \mathcal{A}_0$

$$\psi_{a,t+1} \leq \max_{s^T} \sum_{h=1}^{H} e^h_{t+1} + \sum_{a \in \mathcal{A}_1} d_{a,t+1} K_{a,-1} + A_0 M_t + (A_0 - 1) M^{\psi-}_{a,t+1} = M^{\psi}_{t+1}.$$

Therefore,

$$\Psi_a \left(K^n_{a,t+1}, K^o_{a,t+1} \right) \leq M^{\psi}_{t+1}$$

since

$$\Psi_a \left(K^n_{a,t+1}, K^o_{a,t+1} \right) \leq \psi_{a,t+1}.$$

Also, given

$$K^o_{a,t+1} \leq M^o_{t+1},$$

and Ψ_a is decreasing in $K^o_{a,t+1}$, we have

$$\Psi_a \left(K^n_{a,t+1}, M^o_{t+1} \right) \leq M^{\psi}_{t+1}$$

so

$$K^n_{a,t+1} \leq \Psi^{-1}_{a,2} \left(M^o_{t+1}, M^{\psi}_{t+1} \right) = M^n_{a,t+1}.$$

Finally

$$\begin{aligned}\sum_{h=1}^{H} c_{t+1}^{h} &\leq \sum_{h=1}^{H} e_{t+1}^{h} + \sum_{a\in\mathcal{A}_1} d_{a,t+1}K_{a,-1} + A_0M_t - \sum_{a\in\mathcal{A}_0} \psi_{a,t+1} \\ &\leq \sum_{h=1}^{H} e_{t+1}^{h} + \sum_{a\in\mathcal{A}_1} d_{a,t+1}K_{a,-1} + A_0M_t + \sum_{a\in\mathcal{A}_0} M_{a,t+1}^{\psi-} = M_{t+1}^{c}.\end{aligned}$$

Lastly,

$$L_{a,t+2} \leq \sum_{h=1}^{H} L_a^h = \overline{L}_a$$

and

$$K_{a,t+2}^{f} \leq \sum_{h=1}^{H} k_{a,t+1}^{h} \leq K_{a,t+1}^{n} \leq M_{a,t+1}^{n}$$

therefore

$$y_{t+2} \leq \max_s F_a\left(M_{a,t+2}^{n}, \overline{L}_a, s\right) = M_{a,t+1}^{f}$$

Let

$$M_{t+1} = \max\left(M_{t+1}^{c}, M_{t+1}^{o}, M_{a,t+1}^{n}, M_{a,t+1}^{f}, M_{t+1}^{\psi}, M_{a,t+1}^{\psi-}\right).$$

we have

$$c_t^h, k_{a,t+2}^{h}, K_{a,t+1}^{n}, K_{a,t+1}^{o}, \psi_{a,t+1}, K_{a,t+2}^{f}, y_{a,t+2}$$

are bounded by M_{t+1}.

Step 2 $t = 0$: Similarly proof using (46). ■

Proof of Theorem 2. In this proof, we allow for non-trivial labor choice decision, by supposing that the utility function of each consumer is concave over consumption and leisure $U_h(c,l)$ in Assumption 3b. We restrict choices of produces and consumers to $[-2M_T, 2M_T]$,(keeping bond holding choices in $[-B,+B]$ and labor choices of final good producers in $\left[0, 2\overline{L}_a\right]$) constructed from above. To simplify the proof, we switch from the final good as numeraire to the following normalization:

Let Δ denote the set of prices $(p^c, q_a^*, q_a, d_a, w_a, p_j)$ such that

$$\begin{aligned} p^c, q_a^*, q_a, d_a, w_a, p_j &\geq 0 \\ p^c + \sum_{a\in\mathcal{A}_0} q_a^* + \sum_{a\in\mathcal{A}} q + \sum_{a\in\mathcal{A}_0} d_a + \sum_{a\in\mathcal{A}_0} w_a + \sum_{j\in\mathcal{J}} p_j &= 1\end{aligned}$$

For each state s^t we normalize prices in each time-state pair such that

$$\left(p^c\left(s^t\right), q_a^*\left(s^t\right), q_a\left(s^t\right), d_a\left(s^t\right), w_a\left(s^t\right), p_j\left(s^t\right)\right) \in \Delta.$$

for $t \leq T-1$ and for the final date

$$(q, p^c, d_a, w_a, p) \in \Delta^f,$$

in which

$$\Delta^f = \left\{(p^c, q_a, d_a, w_a) \geq 0 : p^c + \sum_{a\in\mathcal{A}_0} q_a + \sum_{a\in\mathcal{A}_0} d_a + \sum_{a\in\mathcal{A}_0} w_a = 1\right\}.$$

Notice that the no-default constraint has become

$$f_{j,t+1}\left(s^{t+1}\right)=\min\left\{p_t^c\left(s^t\right)b_j\left(s_{t+1}\right),\sum_{a=1}^{A}k_a^j\left(s^t\right)\left(q_a\left(s^{t+1}\right)+d_a\left(s^{t+1}\right)\right)\right\}$$

The optimal decisions of the capital producers yield $\left(K_{a,t}^n, K_{a,t}^o, \psi_{a,t}\right)_{s\in\Sigma^{T-1}}$, and of final good producers yield $\left(K_{a,t}^f, L_{a,t}, y_{a,t}\right)_{s\in\Sigma^T}$ and the decisions of the consumers yields

$$\left(c_t^h, l_{a,t}^h, k_{a,t+1}^h, \phi_{j,t+1}^h\right)_{s\in\Sigma^{T-1}}\times\left(c_t^h, l_{a,t}^h\right)_{s\in\Sigma^T\setminus\Sigma^{T-1}}.$$

Let Z denote the correspondence that maps each set of prices

$$\left(p_t^c, q_t^*, q_t, d_t, w_t, p_{j,t}\right)_{s\in\Sigma^{T-1}}\times\left(q, p_t^c, d_t, w_t\right)_{s\in\Sigma^T\setminus\Sigma^{T-1}}$$

to the excess demand in each market in each time-state pair

$$Z:\Delta^{\left\|\Sigma^{T-1}\right\|}\times\left(\Delta^f\right)^{\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}\rightrightarrows\mathbf{R}^{(1+A_1+4A_0+J)\left\|\Sigma^{T-1}\right\|+(1+3A_0)\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}$$

$$p\in\Delta^{\left\|\Sigma^{T-1}\right\|}\times\left(\Delta^f\right)^{\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}\longmapsto z=(\text{excess demands})\tag{48}$$

The component of the excess demand in each market corresponds to the component of the price system in that market. When $\sigma\in\left\|\Sigma^{T-1}\right\|$ there is one market for final good, A_1 markets for assets with fixed supply and $4A_0$ markets corresponding to new units, old units for asset production, old units for production and labor market for each asset with elastic supply, and finally J market for financial securities. When $\sigma\in\left\|\Sigma^T\right\|$ there are no markets for financial securities nor new units of assets.

In Lemma 7, we establish that Z is upper hemi-continuous and compact, convex-valued. Given each individual choice is bounded, Z is bounded for example by a closed cube K of

$$\mathbf{R}^{(1+A_1+4A_0+J)\left\|\Sigma^{T-1}\right\|+(1+3A_0)\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}.$$

Consider the following correspondence

$$F:\left(\Delta^{\left\|\Sigma^{T-1}\right\|}\times\left(\Delta^f\right)^{\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}\right)\times K\rightrightarrows\left(\Delta^{\left\|\Sigma^{T-1}\right\|}\times\left(\Delta^f\right)^{\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}\right)\times K$$

$$\left\{p\in\Delta^{\left\|\Sigma^{T-1}\right\|}\times\left(\Delta^f\right)^{\left\|\Sigma^T\setminus\Sigma^{T-1}\right\|}, z\in K\right\}\longmapsto\underset{\widetilde{p}\in\left(\Delta^{\left\|s^T\right\|\cdot(T-1)}\times\left(\Delta^f\right)^{\left\|s^T\right\|}\right)}{\arg\max}\left\{\widetilde{p}\cdot z\right\}\times Z(p).$$

Since F is an upper hemi-continous correspondence, with non-empty, compact convex value. Kakutani's theorem guarantees that F has a fixed point

$$\overline{p}=\left(\left(\overline{p}_t^c,\overline{q}_{a,t}^*,\overline{q}_{a,t},\overline{d}_{a,t},\overline{w}_{a,t},\overline{p}_{j,t}\right)_{s\in\Sigma^{T-1}}\times\left(\overline{p}_T^c,\overline{q}_{a,T},\overline{d}_{a,T},\overline{w}_{a,T}\right)_{s\in\Sigma^T\setminus\Sigma^{T-1}},\overline{z}\right).$$

We simplify the notations by denoting

$$\begin{aligned}\overline{p}_t\left(s^t\right) &= \left(\overline{p}^c_t, \overline{q}^*_{a,t}, \overline{q}_{a,t}, \overline{d}_{a,t}, \overline{w}_{a,t}, \overline{p}_{j,t}\right) \text{ for } s^t \in \Sigma^{T-1} \\ \overline{p}_T\left(s^T\right) &= \left(\overline{p}^c_T, \overline{q}_{a,T}, \overline{d}_{a,T}, \overline{w}_{a,T}\right) \text{ for } s^T \in \Sigma^T \backslash \Sigma^{T-1}\end{aligned}$$

Notice that, by summing up over consumers' budget constraint as done in Lemma 6 we obtain the following inequalities

$$\begin{aligned}p^c_t Z^c_t\left(s^t\right) + \sum_{a\in\mathcal{A}_1} q_{a,t} Z^{Ko}_{a,t}\left(s^t\right) + \sum_{j=1}^{J} p_{j,t} Z_{j,t}\left(s^t\right) + \sum_{a\in\mathcal{A}_0} q^*_t Z^{Kn}_{a,t}\left(s^t\right) + \sum_{a\in\mathcal{A}_0} q_{a,t} Z^{K}_{a,t}\left(s^t\right) \\ + \sum_{a\in\mathcal{A}_0} d_{a,t} Z^{Kf}_{a,t}\left(s^t\right) + \sum_{a\in\mathcal{A}_0} w_{a,t} Z^{L}_{a,t}\left(s^t\right) \leq 0 \end{aligned} \tag{49}$$

for each $t \leq T-1$. Notice that for $t = 0$ $\sum_{h=1}^{H} \phi^h_{j,-1} = 0$ and $\sum_{h=1}^{H} k^h_{a,-1} = K_{a,-1}$. For the notations used above, we have

$$Z^c_t\left(s^t\right) = \sum_{h=1}^{H} c^h_t + \sum_{a\in\mathcal{A}_0} \psi_{a,t} - \sum_{h=1}^{H} e^h_t - \sum_{a\in\mathcal{A}_1} d_{a,t} \sum_{h} k^h_{a,t-1} - \sum_{a\in\mathcal{A}_0} y_{a,t} - \sum_{\substack{j=1 \\ f_{j,t}=p^c_t b_{j,t}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t-1}\right) b_{j,t}$$

$$Z^{Kn}_{a,t}\left(s^t\right) = \sum_{h=1}^{H} k^h_{a,t} - K^n_{a,t}$$

$$Z^{Ko}_{a,t}\left(s^t\right) = K^o_{a,t} - \sum_{h=1}^{H} k^h_{a,t-1} - \sum_{\substack{j=1 \\ f_{j,t}<b_{j,t}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t-1}\right) k^a_{j,t}$$

$$Z^{Kf}_{a,t}\left(s^t\right) = K^f_{a,t} - \sum_{h=1}^{H} k^h_{a,t-1} - \sum_{\substack{j=1 \\ f_{j,t}<b_{j,t}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,t-1}\right) k^a_{j,t}$$

$$Z^{L}_{a,t}\left(s^t\right) = L_{a,t} - \sum_{h=1}^{H} l^h_{a,t}$$

For $a \in \mathcal{A}_0$

$$Z^{K}_{a,t}\left(s^t\right) = \sum_{h} k^h_{a,t} - \sum_{h} k^h_{a,t-1} - \sum_{f_{j,t}<p^c_t b_{j,t}} k^j_{a,t} \sum_{h} \phi^h_{j,t-1}$$

and for $j \in \mathcal{J}_t$

$$Z_{j,t}\left(s^t\right) = \sum_{h} \phi^h_{j,t}$$

For $t = T$

$$p^c_T Z^c_T\left(s^T\right) + \sum_{a\in\mathcal{A}_0} q_{a,T} Z^{Ko}_{T}\left(s^T\right) + \sum_{a\in\mathcal{A}_0} d_{a,T} Z^{Kf}_{T}\left(s^T\right) + \sum_{a\in\mathcal{A}_0} w_{a,T} Z^{L}_{T}\left(s^T\right) \leq 0 \tag{50}$$

with

$$Z^c_T\left(s^T\right) = \sum_{h=1}^{H} c^h_T + \sum_{a\in\mathcal{A}_0} \psi_{a,T} - \sum_{h=1}^{H} e^h_T - \sum_{a\in\mathcal{A}_1} d_{a,T} \sum_{h} k^h_{a,T-1} - \sum_{a\in\mathcal{A}_0} y_{a,T} - \sum_{\substack{j=1 \\ f_{j,T}=p^c_T b_{j,T}}}^{J} \left(\sum_{h=1}^{H} \phi^h_{j,T-1}\right) b_{j,T}$$

$$Z_{a,T}^{Ko}\left(s^{T}\right)=K_{a,T}^{o}-\sum_{h=1}^{H}k_{a,T}^{h}-\sum_{\substack{j=1\\f_{j,T}<b_{j,T}}}^{J}\left(\sum_{h=1}^{H}\phi_{j,T-1}^{h}\right)k_{j,T}^{a}$$

$$Z_{a,T}^{Kf}\left(s^{T}\right)=K_{a,T}^{f}-\sum_{h=1}^{H}k_{a,T}^{h}-\sum_{\substack{j=1\\f_{j,T}<b_{j,T}}}^{J}\left(\sum_{h=1}^{H}\phi_{j,T-1}^{h}\right)k_{j,TT}^{a}$$

$$Z_{a,T}^{L}\left(s^{T}\right)=L_{a,T}-\sum_{h=1}^{H}l_{a,T}^{h}$$

We re-write these inequalities compactly as

$$\overline{p}_{t}\left(s^{t}\right)\cdot\overline{z}_{t}\left(s^{t}\right)\leq 0\ \forall t=0,s^{t}.$$

Given

$$\overline{p}_{t}\in\arg\max_{\widetilde{p}\in\Delta}\left\{\widetilde{p}\cdot\overline{z}_{t}\right\}$$

we have (by choosing $\widetilde{p}$ in the corner of Δ or Δ_{f} depending on whether $t<T$) $\overline{z}_{t}\leq 0$ for each time-state pair t,s^{t}. In Lemma 5, the choices are bounded by M_{T} therefore the artificial bound $2M_{T}$ is not binding. Now we can show that prices are strictly positive:
1) $\overline{p}_{t}^{c}>0$ otherwise $\overline{c}_{t}^{h}$ will reach the artificial bound $2M_{T}$, which contradicts the fact that the bound is not binding. Similarly
2) Given $\overline{p}_{t}^{c}>0$, $\overline{d}_{a,t}>0$ otherwise $\overline{K}_{a,t}^{f}$ will reach the artificial bound.
3) Given $\overline{d}_{a,t+1}>0$, $\overline{q}_{a,t}^{*}>0$ (for $a\in\mathcal{A}_{1}$ $\overline{q}_{a,t}^{*}=\overline{q}_{a,t}$) otherwise $\overline{k}_{a,t}^{h}$ will reach the artificial bound.
4) Given $\overline{p}_{t}^{c}>0$, $\overline{q}_{a,t}>0$ otherwise $\overline{K}_{a,t}^{o}$ will reach the artificial bound.
5) If $\overline{w}_{a,t}=0$ then $\overline{L}_{t}=2\overline{L}_{a},\overline{l}_{a,t}^{h}\leq L_{a}^{h}$ which contradicts the negative excess demand in the labor markets, so $\overline{w}_{a,t}>0$.
6) Finally if $\overline{p}_{j,t}^{b}=0$ then $\overline{\phi}_{j,t+1}^{h}=B$ because $f_{j,t+1}>0$, therefore $\sum_{h}\overline{\phi}_{j,t+1}^{h}=HB>0$, which contradicts the negative excess demand in the financial market for asset j.
Therefore, we must have

$$\overline{p}_{t}^{c},\overline{q}_{a,t}^{*},\overline{q}_{a,t},\overline{d}_{a,t},\overline{w}_{a,t},\overline{p}_{j,t}^{b}>0.$$

$\overline{p}_{t}^{c}>0$ also implies budget constraints, and therefore (49) and (50) hold with equality, so markets must clear. The collateral constraints (5) implies that if $\phi_{j,t}<0$ then $-\phi_{jt}<\frac{M_{T}}{k_{j,a}}$, where $k_{j,a}=\min_{s\in\mathcal{S}}k_{j,a}\left(s\right)>\underline{k}$. Therefore if $\phi_{j,t}^{h}>0$; $\phi_{j,t}^{h}<\left(H-1\right)\frac{M_{T}}{k_{j,a}}$. We can choose M_{T} independent of B, so we can choose B such that $B=\left(H-1\right)\frac{M_{T}}{k_{j,a}}$; this artificial constraint will not be binding. To conclude, observing that in this fixed point, all the artificial bounds are slack: we have thus found an equilibrium. ■

Lemma 6 ***(Walras' Law)*** *Given that consumers, firms optimize subject to their constraints, we obtain inequalities (49) and (50).*

Proof. We sum up the budget constraints (37) across all consumers

$$\begin{aligned}
&\sum_h p_t^c c_t^h + \sum_h \sum_{a\in\mathcal{A}_1} q_{a,t} k_{a,t}^h + \sum_h \sum_{a\in\mathcal{A}_0} q_{a,t}^* k_{a,t}^h + \sum_h \sum_{j=1}^J p_{j,t}\phi_{j,t}^h \\
\leq\ & \sum_h p_t^c e_t^h + \sum_h \sum_{a\in\mathcal{A}_0} w_{a,t} l_{a,t}^h + \sum_h \sum_{j=1}^J f_{j,t}\phi_{j,t-1}^h \\
&+ \sum_h \sum_{a\in\mathcal{A}_0} \left(q_{a,t} + d_{a,t}\right) k_{a,t-1}^h + \sum_h \sum_{a\in\mathcal{A}_1} \left(q_{a,t} + d_{a,t}\right) k_{a,t-1}^h \\
&+ \sum_a \Pi_t^f + \sum_a \Pi_t^a
\end{aligned}$$

So, moving endowment in final good e_t^h from the right hand side to the left hand side we obtain

$$\begin{aligned}
& p_t^c \left(\sum_h c_t^h - \sum_h e_t^h\right) + \sum_{a\in\mathcal{A}_1} q_{a,t} \left(\sum_h k_{a,t}^h\right) \\
&+ \sum_{a\in\mathcal{A}_0} q_{a,t}^* \left(\sum_h k_{a,t}^h\right) + \sum_{j=1}^J p_{j,t} \left(\sum_h \phi_{j,t}^h\right) \\
\leq\ & \sum_h \sum_{a\in\mathcal{A}_0} w_{a,t} l_{a,t}^h + \sum_h \sum_{j=1}^J f_{j,t}\phi_{j,t-1}^h \\
&+ \sum_h \sum_{a\in\mathcal{A}_0} \left(q_{a,t} + d_{a,t}\right) k_{a,t-1}^h + \sum_h \sum_{a\in\mathcal{A}_1} \left(q_{a,t} + p_t^c d_{a,t}\right) k_{a,t-1}^h \\
&+ \sum_{a\in\mathcal{A}_0} \Pi_{a,t}^f + \sum_{a\in\mathcal{A}_0} \Pi_{a,t} \qquad (51)
\end{aligned}$$

Notice that

$$\Pi_{a,t}^f = p_t^c y_{a,t} - d_{a,t} K_{a,t}^f - w_{a,t} L_{a,t}$$

and

$$\Pi_{a,t} = q_{a,t}^* K_{a,t}^n - p_t^c \psi_{a,t} - q_{a,t} K_{a,t}^o.$$

and if $f_{j,t} < p_t^c b_{j,t}$

$$f_{j,t} = \sum_{a=1}^A k_{a,t}^j \left(q_{a,t} + d_{a,t}\right).$$

Plugging these equalities into, (51) we obtain exactly the inequality (49). The inequality (50) is obtained similarly. ■

Lemma 7 *Z defined in (48) is upper hemi-continuous and compact, convex-valued.*

Proof. These properties are standard. ■

Proof of Lemma 3. Given any equilibrium, let $\mu_{a,t}$ denote the Lagrange multipliers associated to the collateral constraints, (5) in the consumers' optimization problem. First we show that consumptions are bounded from above and below: Market clearing condition implies $c_t^h \leq \bar{e}$. Second for each t one of the feasible strategies is to consume at least the endowment in each period therefore

$$\sum_{t'=t}^T \mathbf{P}_h \left(s^{t'}|s^t\right) \beta_h^{t'-t} U_h \left(c_t^h\right) \geq \sum_{t'=t}^T \mathbf{P}_h \left(s^{t'}|s^t\right) \beta_h^{t'-t} U_h \left(e_h\left(s_{t'}\right)\right)$$

Therefore

$$U_h\left(c_t^h\right)+\max\left\{\frac{\beta_h}{1-\beta_h}U_h\left(\bar{e}\right),0\right\}\geq\min\left\{\min_{s\in\mathcal{S}}\frac{1}{1-\beta_h}U_h\left(e^h\left(s\right)\right),\min_{s\in\mathcal{S}}U_h\left(e^h\left(s\right)\right)\right\}$$

so

$$c_t^h\geq\underline{c}.$$

Second, we prove by induction that for each $a\in\mathcal{A}_0$, $K_{a,t+1}\leq\overline{K}_a$. Indeed, good market at time t clears implies

$$\Psi_a\left(K_{a,t+1},K_{a,t}^o\right)\leq\bar{e}_a\left(s\right)\leq\Psi_a\left(\overline{K}_a,\overline{K}_a\right)$$

given

$$K_{a,t}^o=K_{a,t}\leq\overline{K}_a$$

and Ψ_a is decreasing in the second parameter, we have

$$\Psi_a\left(K_{a,t+1},K_{a,t}^o\right)\geq\Psi_a\left(K_{a,t+1},\overline{K}_a\right).$$

Therefore

$$\Psi_a\left(K_{a,t+1},\overline{K}_a\right)\leq\Psi_a\left(\overline{K}_a,\overline{K}_a\right).$$

Since Ψ_a is increasing in the first parameters, we have

$$K_{a,t+1}\leq\overline{K}_a.$$

Now, the first-order condition of the asset producers implies

$$q_{a,t}^*=\frac{\partial\Psi_a\left(K_{a,t+1},K_{a,t}^o\right)}{\partial K_{a,t+1}}.$$

Therefore

$$\underline{q}_a^*=\inf_{0\leq K,K^o\leq\overline{K}_a}\frac{\partial\Psi_a\left(K,K^o\right)}{\partial K}\leq q_{a,t}^*\leq\sup_{0\leq K,K^o\leq\overline{K}_a}\frac{\partial\Psi_a\left(K,K^o\right)}{\partial K}=\overline{q}_a^*.$$

Similarly

$$\underline{q}_a=\inf_{0\leq K,K^o\leq\overline{K}_a}-\frac{\partial\Psi_a\left(K,K^o\right)}{\partial K^o}\leq q_{a,t}\leq\sup_{0\leq K,K^o\leq\overline{K}_a}-\frac{\partial\Psi_a\left(K,K^o\right)}{\partial K^o}=\overline{q}_a.$$

The first-order condition with respect to k_{t+1}^h implies

$$\mu_{a,t}^h-q_{a,t}^*U_h'\left(c_t^h\right)+\beta_hE_t^h\left[\left(q_{a,t+1}+d_{a,t+1}\right)U_h'\left(c_{t+1}^h\right)\right]=0$$

therefore

$$\begin{array}{rcl}\overline{q}_a^*U_h'\left(c_t^h\right)&\geq&\beta_hE_t^h\left[\left(q_{t+1}+d_{a,t+1}\right)U_h'\left(c_{t+1}^h\right)\right]\\&>&\beta_hE_t^h\left[d_{a,t+1}U_h'\left(c_{t+1}^h\right)\right]\end{array}$$

so $d_{a,t+1}$ is bounded by

$$\frac{\overline{q}_a^*U_h'\left(\bar{e}\right)}{\beta_h\Pr_h\left(s_{t+1}|s^t\right)U_h'\left(\underline{c}\right)}=\overline{d}_a.$$

Given

$$d_{a,t+1} = F_{a,K}\left(K_{a,t+1}, L_a\right)$$

$K_{a,t+1}$ is bounded from below by $\underline{K}_a > 0$. Also $d_{a,t+1} = F_{a,K}\left(K_{a,t+1}, L_a\right) > F_{a,K}\left(\overline{K}_a, L_a\right) = \underline{d}_a$ Similarly we have bounds $\overline{w}$ and $\underline{w}$ for $w_{a,t+1}$.
For $a \in \mathcal{A}_1$

$$q_{a,t} \leq \frac{H}{K_{a,-1}}\overline{c} = \overline{q}_a$$

otherwise there will be a consumer that holds at least $\frac{K_{a,-1}}{H}$ units of asset a at s^t after paying-off her debt. This consumer can sell part of her holding to pay-off debt and consume the rest of the sale. This strategy would give her more expected utility than her current one. This contradicts the optimally of her current choice. More formally, given

$$\sum_h \left(k_{a,t-1}^h + \sum_j \phi_{j,t-1}^h k_{a,t-2}^j\right) = K_{a,-1},$$

there must exist a consumer h such that

$$k_{a,t-1}^h + \sum_j \phi_{j,t-1}^h k_{a,t-2}^j \geq \frac{K_{a,-1}}{H}.$$

Therefore her budget at the beginning of period t will exceed

$$\begin{aligned}
& e_t^h + \sum_{a\in\mathcal{A}_0} w_{a,t} l_{a,t}^h + \sum_{j=1}^J f_{j,t}\phi_{j,t-1}^h + \sum_{a\in\mathcal{A}}\left(q_{a,t}+d_{a,t}\right)k_{a,t-1}^h \\
\geq\ & \left(q_{a,t}+d_{a,t}\right)\left(k_{a,t-1}^h + \sum_j \phi_{j,t-1}^h k_{a,t-2}^j\right) \\
\geq\ & \left(q_{a,t}+d_{a,t}\right)\frac{K_{a,-1}}{H} \\
>\ & \overline{c}.
\end{aligned}$$

To continue, the first-order condition

$$\mu_{a,t}^h - q_{a,t}U_h'\left(c_t^h\right) + \beta_h E_t^h\left[\left(q_{a,t+1}+d_{a,t+1}\right)U_h'\left(c_{t+1}^h\right)\right] = 0$$

yields

$$q_{a,t} \geq \max_h \frac{\beta_h \min_{s\in\mathcal{S}} d_a(s)\, U_h'\left(\overline{e}\right)}{U_h'\left(\underline{c}\right)} = \underline{q}_a.$$

The first-order condition with respect to $\phi_{j,t+1}$ implies for an agent h with $\phi_{j,t+1}^h \geq 0$ (check the deviation $\phi_{j,t+1}^h + \delta\phi$)

$$-p_{j,t}U_h'\left(c_t^h\right) + \beta_h E_h\left[f_{j,t+1}U_h'\left(c_{t+1}^h\right)\right] \leq 0.$$

So

$$\begin{aligned}
p_{j,t} \ &\geq\ \frac{\beta_h E_h\left[f_{j,t+1}U_h'\left(c_{t+1}^h\right)\right]}{U_h'\left(c_t^h\right)} \\
&\geq\ \frac{\beta_h \min\left(b_j, \underline{k}\left(\underline{q}+\underline{d}\right)\right)U_h'\left(\overline{e}\right)}{U_h'\left(\underline{c}\right)} = \underline{p}_j.
\end{aligned}$$

Moreover we should have

$$\begin{aligned} p_{j,t} &\leq \sum_{a\in\mathcal{A}_0} \overline{q}_a^* k_a^j + \sum_{a\in\mathcal{A}_1} \overline{q}_a k_a^j \\ &\leq \left(\sum_{a\in\mathcal{A}_0} \overline{q}_a^* + \sum_{a\in\mathcal{A}_1} \overline{q}_a\right)\overline{k} = \overline{p}_j, \end{aligned}$$

otherwise it is more than enough to simultaneously buy assets and sell security j, the aggregate demand of ϕ_j will be strictly negative. Also because of the market clearing condition we have $0 \leq k_{a,t+1}^h \leq \overline{K}_a$. Because of the collateral constraint

$$\begin{aligned} \phi_{j,t}^h &\geq \max_a \left(-\frac{\overline{K}_a}{k_{a,t}^j}\right) \\ &\geq \max_a \left(-\frac{\overline{K}_a}{\underline{k}}\right) = \underline{\phi}_j \end{aligned}$$

therefore

$$\phi_{j,t}^h \leq -(H-1)\,\underline{\phi}_j = \overline{\phi}_j.$$

■

Proof of Theorem 2. Let the compact set $\mathcal{T} \subset \widehat{\mathcal{V}}$ denote the set over which the equilibrium endogenous variables of the finite horizon economies lie and E is defined such that the set of equilibrium total units of assets always lie in E as well. For each correspondence $V : \mathcal{S}\times\Omega\times E \rightrightarrows \mathcal{T}$ define an operator that maps the correspondence to a new correspondence $W : \mathcal{S}\times\Omega\times E \rightrightarrows \widehat{\mathcal{V}}$ such that

$$W(s,\omega,K) = \left\{ \begin{array}{c} \widehat{v} \in \mathcal{T} \text{ such that } (s,\omega,K,v) \in \mathcal{X}\text{: } \exists\,(v_s)_{s\in\mathcal{S}} \in g(s,\omega,K,v) \\ \text{such that } \forall s'\ \widehat{v}_{s'} \in V(s',\omega') \text{ in which } v_{s'} = (\omega',K',\widehat{v}_{s'}) \end{array} \right\}$$

Let $V^0 = \mathcal{T}$ and $V^{n+1} = G_{\mathcal{T}}(V^n)$. In Lemma 8 below, we show that V^{n+1} is a non-empty correspondence for all $n \geq 0$. We have $W(s,\omega,K)$ is not empty and $W(s,\omega,K) \subset V^0 = \mathcal{T}$ .It is also easy to show that

$$V(s,\omega,K) \subset V'(s,\omega,K)$$

for all $(s,\omega,K) \in \mathcal{S}\times\Omega\times E$ (denote $V \subset V'$) then the same inclusion holds for W and W'. By definition $V^1 \subset V^0$ so by induction we can show $V^{n+1} \subset V^n$.Therefore we have obtained a sequence of decreasing compact sets. Let

$$V^*(s,\omega,K) = \bigcap_{n=1}^{\infty} V^n(s,\omega,K)$$

Then V^* is a non-empty correspondence and $G_{\mathcal{T}}(V^*) \subset V^*$. Since graph of g is closed, we have that $G_{\mathcal{T}}(V^*)$ is non-empty as well. Let V^* be the 'policy correspondence' and

$$F^*(s,\omega,K,v) = \left\{ \begin{array}{c} (v_s)_{s\in\mathcal{S}} \in g(s,\omega,K,v) \text{ such that} \\ \forall s'\ \widehat{v}_{s'} \in V^*(s',\omega') \text{ where } v_{s'} = (\omega',K',\widehat{v}_{s'}) \end{array} \right\}.$$

Then (V^*,F^*) is a Markov equilibrium. ■

Lemma 8 *V^{n+1} is a non-empty correspondence for all $n \geq 0$.*

Proof. For each n, consider the equilibrium constructed in Lemma 2 for the initial condition (s, ω, K) it is easy to show that the resulting allocation at time 0 belong to $V^n(s, \omega, K)$. For example, for $n = 0$: We use the equilibrium constructed in For each $s_1 \in \mathcal{S}$ Let v_{s_1} is defined by

$$\begin{aligned} q_a^* &= 0 \\ p_j &= 0 \end{aligned}$$

and q_a, w_a, d_a are defined as in that construction. We also add $k_a^h = 0$, $K_a = 0$ and $\phi_j^h = 0$ the other allocations are defined in the construction as well. Then $(s_1, v_{s_1}) \in \mathcal{X}$. It easy to see that $(v_s)_{s \in \mathcal{S}} \in g(s, \omega, K, v)$. Also $\widehat{v}_{s_1} \in V(s_1, \omega_1)$ by definition. ■

Appendix C: Algorithms

Algorithm 1 (Algorithm to Compute Complete Markets Equilibria) *The state space should be*

$$\left((c_h)_{h\in\mathcal{H}}, (K_a)_{a\in\mathcal{A}_0}\right).$$

We find the mapping ρ from that state space into the set of current prices and investment levels $\{q_a, q_a^*, w_a, d_a, K_a^n\}_{a\in\mathcal{A}_0}, \{q_a\}_{a\in\mathcal{A}_1}$, future consumptions $\left\{\left(c_h^+\right)_{h\in\mathcal{H}}\right\}_{s\in\mathcal{S}}$, and $\{p_s\}_{s\mathcal{S}}$ the Arrow-Debreu state prices. There are therefore $5A_0 + A_1 + SH + S$ unknowns.
First, notice that $l_{h,a} = L_a^h$.
Then, for each $a \in \mathcal{A}_0$, from the first order condition for the asset producers and final good producers, we obtain.

$$\begin{aligned} q_a^* &= \frac{\partial \Psi_a(K_a^n, K_a)}{\partial K_a^n} \\ q_a &= -\frac{\partial \Psi_a(K_a^n, K_a)}{\partial K_a} \\ d_a &= F_K\left(K_a, \sum_a l_a^h, s\right) \\ w_a &= F_L\left(K_a, \sum_a l_a^h, s\right) \end{aligned}$$

which give $4A_0$ equations. From the non-arbitrage equations, it should be that

$$q_a^* = \sum_s p_s\left(q_a^+ + d_a^+\right).$$

This gives another A_0 equations.
For each $a \in \mathcal{A}_1$ we also have A_1 equations

$$q_a = \sum_s p_s\left(q_a^+ + d_a^+\right)$$

Regarding p_s, the inter-temporal Euler equation implies

$$p_s = \beta_h \pi^h\left(s, s^+\right)\frac{U_h'\left(c_h^+\right)}{U_h'(c_h)}$$

that give SH equations and finally

$$\sum_h c_h^+ + \sum_a \Psi_a\left(K_a^{n+}, K_a^n\right) = \sum_h e_h + \sum_{a\in\mathcal{A}0} F\left(K_a^n, \sum_a l_a^h\right) + \sum_{a\in\mathcal{A}_1} e_a K_{a,-1}$$

which give other S equations. With these $5A_0 + A_1 + SH + S$ equations, we can solve for the $5A_0 + A_1 + SH + S$ unknowns. That solution determines the mapping $T\rho$.
In order to find an equilibrium corresponding to an initial asset holdings $(\theta_{h,a})_{h\in\mathcal{H},a\in\mathcal{A}}$ we find the value of stream of consumption and endowment of each consumers

$$V_c^h = c_h + \sum_{s\in\mathcal{S}} p_s V_c^{h+}(s)$$

and

$$V_e^h = e_h + \sum_{s\in\mathcal{S}} p_s V_e^{h+}(s)$$

Then we solve for H unknowns $(c_h)_{h\in\mathcal{H}}$ using H equations

$$V_c^h = V_e^h + \sum_{a\in\mathcal{A}} \theta_{h,a} q_a.$$

Remark 4 *When there are no assets with elastic supply, calculation is easier: The state space should be $\left((c_h)_{h\in\mathcal{H}-1}\right)$ We find the mapping ρ from that state space into $\left\{\left(c_h^+\right)_{h\in\mathcal{H}-1}\right\}_{s\in\mathcal{S}}$ future consumptions and $\{p_s\}_{s\mathcal{S}}$ the Arrow-Debreu state prices. In total we have HS unknowns. Notice that we need to keep track of the consumption of only $H-1$ consumers. The consumption of the remaining consumer is determined by the market clearing condition*

$$c_H(s) = e_a(s) - \sum_{h=1}^{H-1} c_h(s).$$

The intertemporal Euler equation implies

$$p_s = \beta_h \pi^h\left(s, s^+\right) \frac{U_h'\left(c_h^+\right)}{U_h'(c_h)}$$

that give HS equations. From these HS equations we can solve for the HS unknowns. When we have CRRA utility functions, we can solve for closed form solutions of p_s and c_h^+.

Algorithm 2 (Algorithm to Compute Incomplete Markets Equilibria) *We look for the equilibrium mapping defined in (25), for each iteration, given ρ^n,*

$$\rho_s^{n+1}(\omega, K_a) = \left(\widehat{v}_{n+1}, \omega_{s,n+1}^+, \mu_{n+1}, \eta_{n+1}\right)$$

is determined to satisfied the following equations

$$\begin{aligned}
0 &= \mu_{a,n+1}^h - q_{a,n+1}^* U_h'\left(c_{n+1}^h\right) \\
&\quad + \beta_h E^h\left\{\left(q_a^+ + d_a^+\right) U_h'\left(c^{h+}\right)\right\} \\
0 &= \mu_{a,n+1}^h \left(k_{a,n+1}^h + \sum_{j\in\mathcal{J}:\phi_j^h<0} k_{a,j}^j \phi_{j,n+1}^h\right) \\
0 &\le k_{a,n+1}^h + \sum_{j\in\mathcal{J}:\phi_j^h<0} k_a^j \phi_{j,n+1}^h.
\end{aligned}$$

The variables with superscript $+$, $q_a^+, d_a^+, c^{h+}, l^{h+}$ *are determined using the mapping* ρ^n *on the state variables* $(s, \omega_s^+, K_{a,n+1})$ *where*

$$K_{a,n+1} = \begin{cases} \sum_{h \in \mathcal{H}} k_{a,n+1}^h & \text{if } a \in \mathcal{A}_0 \\ K_{a,-1} & \text{if } a \notin \mathcal{A}_0 \end{cases}$$

We also require

$$\begin{aligned} 0 &= \sum_{a \in \mathcal{A}} \mu_{a,n+1}^h k_a^j - p_{j,n+1} U_h'\left(c_{n+1}^h\right) \\ &\quad + \beta^h E^h \left\{ f_j^+ U_h'\left(c^{h+}\right)\right\} - \eta_{j,n+1}^h(-) \\ 0 &= -p_{j,n+1} U_h'\left(c_{n+1}^h\right) + \beta^h E^h \left\{ f_j^+ U_h'\left(c^{h+}\right)\right\} + \eta_{j,n+1}^h(-) \\ 0 &= \phi_{j,n+1}^h(+)\,\eta_{j,n+1}^h(+) \\ 0 &= \phi_{j,n+1}^h(-)\,\eta_{j,n+1}^h(-)\,. \end{aligned}$$

The budget constraints of the consumers hold with equality

$$c_{n+1}^h = e^h(s) + \omega^h (q_{n+1} + d_{n+1}) \cdot K - q_{n+1}^* \cdot k_{n+1}^h + w_{n+1} \cdot l_{n+1}^h - p_{n+1} \cdot \phi_{n+1}^h$$

where, for each $a \in \mathcal{A}_0$

$$\begin{aligned} q_{a,n+1}^* &= \frac{\partial \Psi_a (K_{a,n+1}, K_a^o)}{\partial K} \\ q_{a,n+1} &= -\frac{\partial \Psi_a (K_{a,n+1}, K_a^o)}{\partial K^o} \\ d_{a,n+1} &= \frac{\partial F_a (K_a^o, L_{a,n+1})}{\partial K^o} \\ w_{a,n+1} &= \frac{\partial F_a (K_a^o, L_{a,n+1})}{\partial L} \end{aligned}$$

with $l_{a,n+1}^h = L_{h,a}$ *and* $L_{a,n+1} = \sum_{h \in \mathcal{H}} l_{a,n+1}^h$. *Finally, the future wealth distributions are consistent with current asset holdings and future prices*

$$\omega_s^{h+} = \frac{k_{n+1}^h \cdot (q_s^+ + d_s^+) + \sum_{j \in \mathcal{J}} \phi_{j,n+1}^h \min\left\{ b_j(s), \sum_{a \in A} k_{a,n+1}^j (q^+ + d^+)\right\}}{\sum_a (q^+ + d^+) \cdot K_{n+1}}$$

again the variables with superscript $+$, q_s^+, d_s^+, *are determined using the mapping* ρ^n.

Appendix D: One asset economy

Proof of Proposition 8. Since there are only two future states, let u denote the higher return

$$u = \max_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right)\right)$$

and d denote the lower return

$$d = \min_{s^{t+1}|s^t} \left(q\left(s^{t+1}\right) + d\left(s_{t+1}\right)\right)$$

We are considering the set of debt assets that promise 1 in both states and requires k unit of the real asset as collateral. The price of such an asset is p_k

1. $k < \frac{1}{u}$; then this asset is essentially the real asset because its effective pay-off is (ku, kd)
2.$\frac{1}{d} \geq k \geq \frac{1}{u}$. Then the pay-off to the borrower of the asset is

$$(ku - 1, 0)$$

and he has to pay $kq - p_k$: she buys k real asset but she get p_k from selling the financial asset. So the borrowers only choose k such that

$$\frac{ku-1}{kq-p_k}$$

is maximized over $k \in \left[\frac{1}{u}, \frac{1}{d}\right]$. So only assets belonging to

$$\arg\max_{k\in\left[\frac{1}{u},\frac{1}{d}\right]} \frac{ku-1}{kq-p_k} \tag{52}$$

will be chosen by borrowers in equilibrium.
Consider an actively traded financial asset with collateral level k^* belonging to the argmax set above. If another financial asset with $k < k^*$ is also actively traded, price of this asset, p_k, will be strictly less than

$$\frac{ku-1}{k^*u-1}p_{k^*} + \frac{k^*-k}{k^*u-1}q.$$

Otherwise, due to collateral value of the real asset, buyers of this asset will strictly prefer the portfolio $\frac{ku-1}{k^*u-1}$ units of financial asset k^* and $\frac{k^*-k}{k^*u-1}$ units of the real asset. This portfolio gives the same payoff value as buying one unit of financial asset k because

$$\begin{bmatrix} 1 \\ kd \end{bmatrix} = \frac{ku-1}{k^*u-1}\begin{bmatrix} 1 \\ k^*d \end{bmatrix} + \frac{k^*-k}{k^*u-1}\begin{bmatrix} u \\ d \end{bmatrix},$$

on top of that it gives the buyer an additional collateral value from holding the real asset.
Therefore

$$\frac{ku-1}{kq-p_k} < \frac{k^*u-1}{k^*q-p_{k^*}}.$$

Thus every seller of this asset k will strictly prefer selling asset k^*. ■

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