Amplification and Asymmetric Effects without Collateral Constraint*

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Abstract

The seminal contribution of Kiyotaki and Moore (1997) has spurred a vast literature on the importance of collateral constraint in propagating and amplifying shocks to the economy. However, collateral constraint implicitly assumes state-incontingent debt, i.e., markets are incomplete. It is possible that a large part of the amplification effect of collateral constraint is actually due to market incompleteness. We study a simple, calibrated model and solve it with and without collateral constraint and find that indeed market incompleteness by itself plays a quantitatively significant role in the amplified and asymmetric responses of the economy to exogenous shocks.

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1 Introduction

The seminal contribution of Kiyotaki and Moore (1997) has spurred a vast literature on the importance of collateral constraint in propagating and amplifying shocks to the economy including Iacoviello (2005) and Mendoza (2010). However, more recent important papers on the same topic, Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) obtain the effects without imposing any collateral constraint. The common feature of these papers is that agents borrow through state-incontingent debt, i.e., markets are incomplete. It is possible that a large part of the amplification effect of collateral constraint is actually due to market incompleteness. In this paper, we build a simple, calibrated model and solve it with and without collateral constraint in order to understand the qualitative and quantitative importance of market incompleteness relative to collateral constraint.

The model is a simplified version of Iacoviello (2005). We have two types of risk-averse agents, the entrepreneurs and the households in a one-consumption good economy with fixed supply of an asset called land. There is a land market in which all agents participate. The entrepreneurs can combine land with labor supplied by the households to produce consumption good. This production process is subject to aggregate stochastic productivity shocks. The households earn wages from working for the entrepreneurs and decide how much to consume in consumption good and in land. We assume that the entrepreneurs are less patient than the households, and thus they tend to borrow from the households. In this economy, we consider three alternative financial market structures. In the first one, the benchmark model - model with incomplete markets - the entrepreneurs can borrow from the households using state-incontingent debt only subject to the non-Ponzi condition. In the second model - model with collateral constraint - debt is still state-incontingent but the entrepreneurs are subject to the collateral constraint, i.e., borrowing is less than a fraction of the expected value (or current value) of the entrepreneurs’ land holding. In the last model - model with collateral constraint and complete markets - the entrepreneurs can sell a complete set of state-contingent securities to the households subject to a collateral constraint on each state-contingent security. We use the Markov equilibrium definition and global nonlinear method developed by Kubler and Schmedders (2003) and Cao (2010) to solve for the equilibrium in each model.

We find that, in the benchmark incomplete markets model, the equilibrium dynamics exhibits amplification and asymmetric responses to symmetric exogenous productivity shocks. For example, land price and output increase after a good shock by less than they decrease after a bad shock. The amplification and asymmetric effects in this model are
due to the net worth effect as follows. An initial negative shock decreases the net worth of the levered entrepreneurs. Due to risk-aversion, these entrepreneurs try to smooth consumption but their debt is not state-contingent so they have to liquidate some of their land holding to maintain a certain level of consumption. Their selling activities depress the price of land, and further lowering their net worth, setting off the vicious circle of falling land price and falling net worth. At the heart of this vicious circle is the pecuniary externality due to market incompleteness a la Geanakoplos and Polemarchakis (1986), i.e., when selling off some of their land holding, each entrepreneur does not take into account the negative effect of falling land price on the net worth of other entrepreneurs.

In the second model with collateral constraint, similar to the findings in the collateral constraint literature, the dynamics of the economy can be divided into two regions. In the first region, the collateral constraint is not binding and in the second region, the collateral constraint is binding or close to binding. In the second region, land price and output of the economy are much more sensitive to changes in net worth of the entrepreneurs. The economy also exhibits asymmetric responses to exogenous productivity shocks. We also observe that the probability that we are in the second region decreases rapidly in the size of the shocks due to the precautionary saving motive of the entrepreneurs.

In both incomplete markets and collateral constraint models, the economy exhibits amplified and asymmetric responses to symmetric exogenous shocks. Quantitatively, the responses are only slightly smaller in incomplete markets model compared to the collateral constraint model. These results suggest that market incompleteness alone accounts for a significant part of the responses to shocks in the benchmark model.

To further understand this point, we consider the equilibrium in the last model with collateral constraint and complete markets, in which the entrepreneurs have access to a complete set of state-contingent securities but the sale of these securities has to be collateralized by land. In the long run the economy converges to a single level of wealth distribution, i.e., we have some sort of dynamically complete insurance. At this level of wealth distribution, there is no amplification nor asymmetry effects of exogenous shocks to the economy. These results demonstrate that collateral constraint has to be coupled with markets incompleteness in order to generate significant amplification and asymmetric responses to shocks.

The main results of the paper can be summarized by Table 1. This table shows the average changes in land price and output, conditional on the current normal state after a 3% positive productivity shock (columns 2 and 6, next period state is expansion) and after a 3% negative productivity shock (columns 3 and 5, next period state is recession) under

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1Brunnermeier and Sannikov (2014) call this channel amplification through prices.
Table 1: Average land price and output changes in normal state, 3% shock

<table>
<thead>
<tr>
<th>Type of constraint</th>
<th>Land price</th>
<th>Output</th>
<th>Land price</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Recession</td>
<td>Expansion</td>
<td>Recession</td>
</tr>
<tr>
<td>incomplete markets</td>
<td>3.21%</td>
<td>-3.45%</td>
<td>3.25%</td>
<td>-3.44%</td>
</tr>
<tr>
<td>collateral constraint (IM)</td>
<td>3.30%</td>
<td>-3.76%</td>
<td>3.15%</td>
<td>-3.50%</td>
</tr>
<tr>
<td>collateral constraint (alt, IM)</td>
<td>3.32%</td>
<td>-3.83%</td>
<td>3.17%</td>
<td>-3.58%</td>
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<tr>
<td>collateral constraint (CM)</td>
<td>3.01%</td>
<td>-2.99%</td>
<td>2.97%</td>
<td>-2.97%</td>
</tr>
<tr>
<td>complete markets</td>
<td>3.00%</td>
<td>-3.00%</td>
<td>2.97%</td>
<td>-2.97%</td>
</tr>
</tbody>
</table>

different financial market structures - benchmark model with incomplete markets (row 3), collateral constraint models (rows 4 and 5), collateral constraint with complete markets (row 6), and complete markets (row 7) - with the main parameters calibrated to the U.S. economy from Iacoviello (2005). We compare the responses of land price and output to shocks under different financial markets structures to the responses in the model with complete markets (without collateral constraint). First, Table 1 shows the asymmetric effect under both incomplete markets and collateral constraint, i.e. good shocks increase land price and output by less than bad shocks decrease land price and output. But the asymmetric effect is absent under collateral constraint with complete markets. Second, amplification effect is present under both incomplete markets and collateral constraint, i.e. land price and output change by more than 3% after the shock, even though the effect is slightly weaker under incomplete markets. Amplification effect is also absent under collateral constraint with complete markets.

The paper is related to the vast literature on the effects of collateral constraint other than the papers cited above. In particular, the benchmark model is a simplified version of Iacoviello (2005). But instead of relying on log-linearization as in Iacoviello (2005), we solve for the global nonlinear dynamics of the equilibrium. To do so, we use the concept of Markov equilibrium and the numerical method developed by Kubler and Schmedders (2003) and Cao (2010) but extend it to a production economy with elastic labor supply, housing consumption, and with natural borrowing limit. This global nonlinear solution

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2 Land price and output stay almost the same if the economy stays in the normal state.
3 Output here is defined as total amount of consumption good produced by the entrepreneurs using land and labor. The definition omits the imputed rental value of land from the housing consumption of the households. The same results hold, however, when we add this imputed rental value of land to the current definition of output.
4 The responses are the averages over the stationary distribution, but if we condition on lower values of the entrepreneurs’ wealth, the responses are much larger.
5 The magnitude of amplification is similar to the one in Iacoviello (2005).
6 Iacoviello (2014) suggests a way to adapt the log-linearization method in a piecewise fashion to handle occasionally binding constraints.
7 Indeed, with housing as durable good, we need to make a change to the timing of production com-
also allows us to quantitatively assess the accuracy of the log-linearization solution. Another methodological contribution of this paper is that we extend the numerical method Kubler and Schmedders (2003) and Cao (2010) to allow for a wide range of financial markets structure including incomplete markets with exogenous borrowing constraints, and collateral constraint with complete markets.

In a small open economy framework, Mendoza (2010) also compares the equilibrium under collateral constraint versus the equilibrium under an exogenous borrowing constraint limit and finds that exogenous borrowing limit weakens the amplification effect on Tobin’s Q by a factor of 5.75. The difference between our results and his comes from the fact that the supply of the collateral asset (capital) is elastic in Mendoza (2010), while it is completely inelastic in our model. Inelastic supply of asset implies more volatility in the price of asset and gives more room for negative shocks to be amplified by the selling activities of the constrained agents, even in the absence of collateral constraint. Moreover, Appendix B shows that imposing exogenous borrowing constraint reduces significantly the amplification and asymmetric effects of incomplete markets.

The paper is organized as follows. Section 2 presents the benchmark incomplete markets model and the solution method, as well as reasonable parameters to analyze the solution of this benchmark model. Section 3 studies the collateral constraint model and compare it to the benchmark model. Section 4 studies the complete markets model with collateral constraint and also compare it to the collateral constraint model. Section 5 concludes. Other proofs and constructions are presented in the appendix.

## 2 Benchmark model

The model is based on Iacoviello (2005). It is one of simplest models of a production economy in which a durable asset (land) is used as collateral to borrow and as an input in production. Moreover the model is calibrated to the U.S. economy. The solution method, intuitions, and results based on this model should carry over to similar models.

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8 Michele Boldrin and Fisher (2001) show that when capital supply is flexible, it is impossible to match the observed volatility and the equity premium in equity prices in the U.S.

9 Cordoba and Ripoll (2004) is another simple extension of Kiyotaki and Moore (1997) to nonlinear production and concave utility functions. In Appendix D, we show that the solution method used in this paper applies to this model as well. Despite its simplicity, the model is not calibrated to the U.S. data, thus is not sufficient to make a quantitative point.
2.1 Economic environment

Consider an economy inhabited by two types of agents: entrepreneurs and households who are both infinitely lived and of measure one. There is one consumption good. Entrepreneurs produce the consumption good by hiring household labor and combining it with land. Households consume the consumption good and land (housing), and supply labor to the entrepreneurs.

We adopt the standard notation of uncertainty. Time is discrete and runs from 0 to infinity. In each period, an aggregate shock $s_t$ is realized. We assume that $s_t$ follows a finite-state Markov chain. Let $s^t = (s_0, s_1, ..., s_t)$ denote the history of realizations of shocks until date $t$. To simplify the notation, for each variable $x$, we use $x_t$ as a shortcut for $x_t(s^t)$.

Households maximize a lifetime utility function given by

$$
E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \frac{c_t^1}{1 - \sigma_2} - 1 \right) + \frac{j^1}{1 - \sigma_h} - 1 - \frac{1}{\eta} (L_t^1)^\eta \right\},
$$

where $E_0 [.]$ is the expectation operator, $\beta \in (0, 1)$ is the discount factor, $c_t^1$ is consumption at time $t$, $h_t^1$ is the holding of land. $L_t^1$ denotes the hours of work. Households can trade in the market for land as well as a state-incontingent bond market. The budget constraint of the households is

$$
c_t^1 + q_t (h_t^1 - h_{t-1}^1) + p_t b_t^1 \leq b_{t-1}^1 + w_t L_t^1.
$$

Given land in the utility function of households, we have implicitly $h_t^1 \geq 0$.

Entrepreneurs use a Cobb-Douglas constant-returns-to-scale technology that uses land and labor as inputs. They produce consumption good $Y_t$ according to

$$
Y_t = A_t h_t^v L_t^{1-v},
$$

where $A_t$ is the aggregate productivity which depends on the aggregate state $s_t$, $h_t$ is real estate input, and $L_t$ is labor input.

In contrast to Iacoviello (2005), the production function uses the contemporaneous land holding of the entrepreneurs instead of the land holding from the previous period. This minor modification turns out to be crucial to apply the concept of Markov equilibrium in Subsection 2.2 and the solution method in Subsection 2.3.2. However this modification does not affect key economic forces.

\[^{10}\text{Given their higher discount factor, the households tend to lend to the entrepreneurs so we do not need to impose any borrowing constraint on the households.}\]
We want the entrepreneurs to borrow from the households so we assume that the entrepreneurs discount the future at the rate $\gamma < \beta$. The entrepreneurs maximize

$$E_0 \sum_{t=0}^{\infty} \gamma^t \left( c_t \right)^{1-\sigma_1} \frac{1}{1-\sigma_1}$$  \hspace{1cm} (4)$$

subject to the budget constraint

$$c_t + q_t (h_t - h_{t-1}) + p_t b_t \leq b_{t-1} + Y_t - w_t L_t,$$  \hspace{1cm} (5)$$

and the collateral constraint (13). Output $Y_t$ is produced by combining land and labor using the production function (3). Given the production function of the entrepreneurs, we have implicitly $h_t \geq 0$.

In this benchmark model, we do not impose collateral constraint as in Iacoviello (2005), so we need to implicitly impose no-Ponzi scheme conditions on the entrepreneurs and households, i.e.,

$$\lim_{t \to \infty} \left( \prod_{t'=0}^{t-1} p_{t'} \right) b_t \leq 0$$

and

$$\lim_{t \to \infty} \left( \prod_{t'=0}^{t-1} p_{t'} \right) b'_t \leq 0.$$

To finish the description of the model, here we assume that the only source of uncertainty is the aggregate productivity $A_t$. It is straightforward to extend the model to incorporate other sources of uncertainty such as uncertainty in housing preference $j$.

### 2.2 Equilibrium

The definition of the sequential competitive equilibrium for this economy is standard.

**Definition 1.** A competitive equilibrium is sequences of prices $\{p_t, q_t, w_t\}_{t=0}^{\infty}$ and allocations $\{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t\}$ such that (i) the $\{c'_t, h'_t, b'_t, L'_t\}$ maximize (1) subject to budget constraint (2) and the no-Ponzi condition and $\{c_t, h_t, b_t, L_t\}$ maximize (4) subject to budget constraint (5) and the no-Ponzi condition, and production technology (3) given $\{p_t, q_t, w_t\}$ and initial asset holdings $\{h_{-1}, b_{-1}, h'_{-1}, b'_{-1}\}$; (ii) land, bond, labor, and good markets clear: $h_t + h'_t = H$, $b_t + b'_t = 0$, $L_t = L'_t$, $c_t + c'_t = Y_t$. 

7
Let $\omega_t$ denote the normalized financial wealth of the entrepreneurs:

$$\omega_t = \frac{q_t h_{t-1} + b_{t-1}}{q_t H},$$

(6)

and $\omega'_t$ denote the normalized financial wealth of the households:

$$\omega'_t = \frac{q_t h'_{t-1} + b'_{t-1}}{q_t H}.$$

By the housing and bond market clearing conditions, we have $\omega'_t = 1 - \omega_t$ in any competitive equilibrium. Therefore in order to keep track of the normalized financial wealth distribution between the entrepreneurs and the households, $(\omega_t, \omega'_t)$, in equilibrium, we only need to keep track of $\omega_t$. To simplify the language, we use the term wealth distribution for normalized financial wealth distribution.

Following Kubler and Schmedders (2003) and Cao (2010), we define Markov equilibrium as follows.

**Definition 2.** A Markov equilibrium is a competitive equilibrium in which prices and allocations at time $t$, as well as the wealth distribution at time $t+1$ under different realizations of the exogenous shocks $s_{t+1}$ depend only on the wealth distribution at time $t$, $\omega_t$ as well as the exogenous state $s_t$.

This Markov equilibrium definition features endogenous state variable $\omega_t$ that depends on land price $q_t$ (which by itself depends on the state variable). This equilibrium was first studied in Duffie, Geanakoplos, Mas-Colell, and McLennan (1994). Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013) use the same type of equilibrium definition in their continuous time models. We are going to use the algorithm developed in Kubler and Schmedders (2003) and Cao (2010) to compute this Markov equilibrium.

### 2.3 Solution

In this Subsection, we first show the equations that characterize a competitive equilibrium. In the absence of borrowing constraints, the model does not have a steady-state in the absence of uncertainty because of the differences in the discount factors of the households and the entrepreneurs. However, out of steady state, uncertainty prevents the entrepreneurs from borrowing too much because of the precautionary saving motive. Thus, Markov equilibrium exists with globally bounded amount debt held by the entrepreneurs.
In later sections, when we introduce borrowing constraints, either endogenous collateral constraint or exogenous borrowing limit, a steady-state exists and Markov equilibrium converges to the steady state when uncertainty vanishes.

### 2.3.1 Equilibrium equations

Given their housing holding at time $t$, $h_t$, the entrepreneurs choose labor demand $L_t$ to maximize profit

$$\max_{L_t} \{ Y_t - w_t L_t \}$$

subject to the production technology given in (3). The first order condition (F.O.C) with respect to $L_t$ implies

$$w_t = (1 - \upsilon) A_t h_t^{\upsilon} L_t^{-\upsilon},$$

i.e. $L_t = \left( \frac{(1-\upsilon)A_t}{w_t} \right)^{\frac{1}{\upsilon}} h_t$ and profit

$$Y_t - w_t L_t = \pi_t h_t$$

where $\pi_t = \upsilon A_t \left( \frac{(1-\upsilon)A_t}{w_t} \right)^{\frac{1-\upsilon}{\upsilon}}$ is profit per unit of land.

The first-order conditions with respect to $h_t$ and $b_t$ in the maximization problem of the entrepreneurs imply

$$(\pi_t - q_t) c_t^{-\sigma_1} + \gamma E_t [q_{t+1} c_{t+1}^{-\sigma_1}] = 0 \quad (8)$$

and

$$- p_t c_t^{-\sigma_1} + \gamma E_t \left[ c_t^{-\sigma_1} \right] = 0. \quad (9)$$

Similarly, the F.O.Cs for households are

$$h_t' : -q_t c_t'^{-\sigma_2} + j h_t'^{-\sigma_6} + \beta E_t \left[ q_{t+1} c_{t+1}'^{-\sigma_2} \right] = 0 \quad (10)$$

$$b_t' : -p_t c_t'^{-\sigma_2} + \beta E_t \left[ c_{t+1}'^{-\sigma_2} \right] = 0 \quad (11)$$

$$L_t' : w_t c_t'^{-\sigma_2} = L_t^{\eta - 1} \quad (12)$$

The first order conditions with respect to $h_t$ and $h_t'$ shed light on the determinants of land price. We rewrite (8) as
\[ q_t = \pi_t + \gamma E_t[q_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma_1}] \]

\[ = E_t\left[\sum_{s=0}^{\infty} \gamma^s \left\{ \prod_{r=0}^{s-1} \left(\frac{c_{t+r+1}}{c_{t+r}}\right)^{-\sigma_1}\right\} \pi_{t+s}\right]. \]

The right hand side of this equation show that, from the entrepreneurs point of view, land price is the net present discounted value of present and future profit from production using land and the discount factor depends on the marginal utility of the entrepreneurs. Similarly, we re-write (10) as

\[ q_t = \frac{j(h_t')^{-\sigma_h}}{c_t'} + \beta E_t\left[ q_{t+1}\left(\frac{c_{t+1}'}{c_t'}\right)^{-\sigma_2}\right] \]

\[ = E_t\left[\sum_{s=0}^{\infty} \beta^s \left(\frac{c_{t+s}'}{c_t'}\right)^{-\sigma_2} j(h_{t+s}')^{-\sigma_h}\right]. \]

From the point of view of the households, house price is the present discounted value of current and future marginal utility from housing.

Despite the fact that we do not impose any constraint on the entrepreneurs’ borrowing except for the no-Ponzi condition, the following lemma shows that, in equilibrium, the financial wealth of the entrepreneurs is endogenously bounded from below.

**Lemma 1.** *In any competitive equilibrium, thus any Markov equilibrium, we must have* \( \omega_t \geq 0 \)* *for all* \( t \) *and* \( s' \).

**Proof.** We prove this result by contradiction. Suppose that in a competitive equilibrium, there is \( t \) and \( s' \) such that \( \omega_t(s') < 0 \). Given the formula for the profit maximization of the entrepreneurs above and the definition of financial wealth \( \omega_t \), the budget constraint (5) can be re-written as

\[ c_t + (q_t - \pi_t) h_t + p_t b_t \leq q_t \omega_t. \]

Pick a \( \lambda > 1 \), and consider an alternative trading and consumption plan \( \{\tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'}\}_{t'=0}^{\infty} \) for the entrepreneurs which is the same as the initial plan for \( t' < t \) but for \( t' \geq t \):

\[ \{\tilde{c}_{t'}, \tilde{h}_{t'}, \tilde{b}_{t'}\}_{t'=t+1}^{\infty} = \{\lambda c_{t'}, \lambda h_{t'}, \lambda b_{t'}\}_{t'=t+1}^{\infty} \]
and
\[
\begin{align*}
\tilde{c}_t &= \lambda c_t - (\lambda - 1)q_t \omega_t > \lambda c_t \\
\tilde{h}_t &= \lambda h_t \\
\tilde{b}_t &= \lambda b_t.
\end{align*}
\]

This alternative plan \( \{\tilde{c}_t', \tilde{h}_t', \tilde{b}_t'\}_{t'=t+1}^\infty \) clearly deliver strictly higher utility to the entrepreneurs while satisfies all the constraints, including the no-Ponzi condition. This contradicts the fact that the initial plan is optimal. Therefore \( \omega_t \geq 0 \) for all \( t \) and \( s^t \).

We interpret this lower bound of the entrepreneurs’ wealth as their natural borrowing limit.

### 2.3.2 Global nonlinear method

We also solve the exact nonlinear equilibrium of the model using the algorithm in Kubler and Schmedders (2003) and Cao (2010). In particular, we solve for Markov equilibrium in this economy. The original algorithm in Kubler and Schmedders (2003) is for endowment economy. Cao (2010) extends this algorithm to a production economy with capital accumulation. In the current paper, we show that the original algorithm works similarly when we add labor choice as well as housing consumption decision of the households.

Our algorithm looks for a Markov equilibrium mapping from the financial wealth distribution, \( \omega_t \) - defined in (6), and aggregate shock, \( s_t \), to land price, \( q_t \) and bond price, \( p_t \), the allocation \( \{c_t, h_t, b_t, c_t', h_t', b_t', L_t'\} \) and wage \( w_t \), as well as future financial wealth distribution, \( \omega_{t+1} \), depending on the realization of future aggregate shocks. Indeed given the mapping from \( \omega_{t+1} \) to \( \{q_{t+1}, c_{t+1}, c_{t+1}'\} \), for each \( \omega_t \) and \( s_t \), we can solve for \( \{c_t, h_t, b_t, c_t', h_t', b_t', L_t'\} \) and \( \omega_{t+1} \) using the equations (8), (9), (10), (11), (12), the housing and bond market clearing conditions, as well as the future financial wealth distribution for each future state. Here, we follow the procedure in Cao (2010), instead of the one in Kubler and Schmedders (2003), in solving for \( \omega_{t+1} \) simultaneously with other unknowns. The additional equations needed to solve for \( \omega_{t+1} \) are equation (6) applied to each of the future state \( s_{t+1} \): \( \omega_{t+1} = \frac{q_{t+1}(\omega_{t+1},s_{t+1})h_t+b_t}{q_{t+1}H} \), in which the mapping from future wealth distribution and exogenous state to land price, \( q_{t+1}(\omega_{t+1},s_{t+1}) \), is determined from the previous iteration of the algorithm. It is easy to verify that the number of unknowns are exactly the same as the number of equations.

The algorithm starts by solving for the equilibrium mapping for 1-period economy. Then given the mapping for \( T \)-period economy (from period 0 to 1), we can solve the
mapping for \((T + 1)\)-period economy following the procedure described above. The algorithm converges when the mappings for \(T\)-period economy and \((T + 1)\)-period economy are sufficiently close to each other.

An important difference relative to Kubler and Schmedders (2003) and Cao (2010) is that there is not any borrowing constraint on the entrepreneurs, therefore when \(\omega_t\) is sufficiently low, \(c_t = 0\) and the first-order conditions (8) and (9) are not well-defined. To deal with this issue, we look for the threshold \(\omega_t\) such that at \(\omega_t = \omega_t\), \(c_t = \xi > 0\), \(\xi\) is small and predetermined. At \(\omega_t\), we know \(c_t = \xi\), so we solve for \(\{\omega_t, h_t, b_t, c_t, h_t', b_t', L_t, \omega_{t+1}\}\) given the same set of equations described above. Lemma 1 shows that \(\omega_t > 0\). Numerically, when \(\xi\) is close to zero, \(\omega_t\) is also close to zero.\(^{11,12}\) The lower bound \(\omega_t\) should depend on exogenous shock \(s_t\), as a result, in contrast to the standard algorithm in Kubler and Schmedders (2003) and Cao (2010), the grid for \(\omega_t\), \([\omega_t, 1]\), depends on the exogenous state \(s_t\), as well as on the horizon of the approximate finite-horizon economy.\(^{13}\) With the calibrated parameters we use in this paper, the stationary distribution also concentrates around very low levels of \(\omega\) (around 0.02), so when we discretize \([\omega_t, 1]\), we put more points in the range of low values of \(\omega\).

### 2.4 Parameter values

We use parameter values from Iacoviello (2005), given in Table 2. In particular, the value of \(\nu\) is chosen to make sure that the value of land holding for entrepreneurs (commercial real estate in the data) in steady state is around 20%.

In order to use the global nonlinear method in Subsection 2.3.2, we need to discretize the process for \(A_t\) by finite number of points. For \(A_t\) we use a three point process, \(A_t \in \{A - \Delta, A, A + \Delta\}\), which corresponds to booms, \(s_t = G\), normal times, \(s_t = N\), and recessions, \(s_t = B\).\(^{14}\) We assume the following form of transition matrix

\[
\Pi = \begin{bmatrix}
\pi & 1 - \pi & 0 \\
\frac{1 - \pi_0}{2} & \pi_0 & \frac{1 - \pi_0}{2} \\
0 & 1 - \pi & \pi \\
\end{bmatrix}.
\]

\(^{11}\)When the entrepreneurs have some labor endowment, \(\omega_t\) can actually be slightly negative.

\(^{12}\)During iterations, when \(\omega_t \leq \omega_t\), we extrapolate the functions \(x(\omega_t, s_t)\), where \(x = q, c, c'\), to obtain the values of \(x\) below \(\omega_t\).

\(^{13}\)Given households are allowed to borrow from entrepreneurs, the upper bound for \(\omega_t\) should exceed 1. However, around the steady state, \(\omega_t\) tends to fall below 1, because of the consumers’ tendency to lend given their higher discount factor.

\(^{14}\)Two-state process is enough to illustrate the amplification and asymmetric effects, but in order to match several moments of the productivity process in the U.S. we need at least three states.
Table 2: Baseline Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tr>
<td>$A$</td>
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</tr>
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<td>$\sigma_h$</td>
<td>1.01</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.01</td>
</tr>
<tr>
<td>$\upsilon$</td>
<td>0.03</td>
</tr>
</tbody>
</table>

The exogenous stochastic process of productivity is totally symmetric. However, due to collateral constraint and incomplete markets, the resulting dynamics of the economy become asymmetric as shown in Subsection 2.5 below.

The values of $\pi_0$ and $\pi$ are calibrated using history data in the United States. According to the definitions of business cycle expansions and recessions by NBER, there are 12 recessions in total from post-WWII (1945) to December 2013, with average length of each recession as 3.6 quarters. The share of months spent in crisis is 15.7% in total.

Given the transition matrix $\Pi$ above, we have the average length of each recession is 

$$\frac{1}{1-\pi}$$

so $\pi = 72.04\%$. $\pi_0$ is chosen such that the probability of a recession in the stationary distribution for $A_t$, i.e., $A_t = A - \Delta$ matches the share of months spent in crisis which implies $\pi_0 = 87.2\%$.\(^{15}\)

As we normalize $A$ to 1, in numerical simulations, we vary $\Delta$ from 1% to 5%, in order to study the non-linear effects of large shocks. However, in benchmark set of parameters, we choose, $\Delta = 3\%$ in order to match the standard deviation of productivity in the U.S. economy of about 14%, which is used by Khan and Thomas (2013).\(^{16}\)

2.5 Numerical results

In this Subsection, we present the numerical results for the benchmark incomplete markets economy with the calibrated parameters in Subsection 2.4.

The key feature of the solution method presented in Subsection 2.3.2 is to solve for the

\(^{15}\)From the transition matrix for $A_t$, the probability of recession in the stationary distribution is 

$$\frac{1}{1-\pi}$$

\(^{16}\)Given the exogenous process for productivity, the standard deviation of productivity is given by 

$$\sqrt{\frac{1-\pi_0}{1-\pi-\pi_0}\Delta}$$

When $\Delta$ is close to 3% and $\pi$ and $\pi_0$ are chosen in the text and the standard deviation of productivity is around 15% as in the data.

13
endogenous lower bound $\omega_t$ as defined in Subsection 2.3.2 in each iteration.\footnote{This method also applies to the case with strictly positive labor endowment for the entrepreneurs. In this case $\omega_t$ are negative.} When we solve for $T$–period economy, the lower bound is decreasing in $T$ and approaches 0 from above as $T$ goes to infinity. Figure 1 shows how the lower bound $\omega_t$ changes over time and across states $s_t$ (thick blue lines for good state $s_t = G$, dashed purple lines for normal state $s_t = N$, and dotted red line for bad state $s_t = B$). The lower bounds are lower under good state than under bad state, which is intuitive because good state leads to high profit of the entrepreneurs so can borrow more from the households.

Another important ingredient of the solution method is for each $(s_t, \omega_t)$, we have to solve for future wealth distribution $\omega_{t+1}$ for each realization of $s_{t+1}$. The left panel of Figure 2 shows $\omega_{t+1}$ as functions of $\omega_t$ and $s_{t+1}$ given that $s_t = N$. Given that the entrepreneurs are more exposed to productivity shock through production, their wealth increases (relative the households’) as good shock hits next period (solid blue line), and decreases as bad shock hits next period (dotted red line). If $s_{t+1}$ stays at the normal state, then wealth distribution remains almost unchanged as the future wealth function (dashed purple line) stays close to the 45° line (dashed black line). The transition functions for wealth distribution combined with the transition matrix of the exogenous states determines the stationary wealth distribution in the right panel (we plot the density of the distribution). Given the small share of land in the aggregate production function, the wealth share of the entrepreneurs always stay below 10% in the steady state.
Figure 2: Transition and stationary distribution of wealth

Figure 3 shows the policy (consumption of the entrepreneurs and the households and aggregate output) and pricing functions (land price) conditional on the exogenous state $s_t$ and the endogenous state $\omega_t$. Even though the global nonlinear methods solves for the policy and pricing functions for the whole range of $\omega_t$, Figure 3 are restricted to the values of $\omega_t$ in the support of the stationary distribution of $\omega_t$. We observe that, despite the absence of collateral constraint, land price and output functions are nonlinear in wealth distribution. In particular they are more sensitive to changes in wealth distribution when the wealth of the entrepreneurs is low.

Nonlinearity implies asymmetric responses of equilibrium land price and output with respect to productivity shocks. Starting from $s_t = N$, a good shock, i.e., $s_{t+1} = G$ increases the entrepreneurs’ wealth and bad shock $s_{t+1} = B$ decreases the entrepreneurs’ wealth as shown in Figure 2. But conditional on the same change in entrepreneurs’ wealth, land price and output increase after a good shock by less that they decrease after a bad shock due to nonlinearity. This leads to the asymmetric responses of equilibrium land price and output to symmetric shocks, as shown quantitatively in Table 1.

The asymmetric responses come from the net worth effect as follows. Initially, a neg-

---

$^{18}$A more precise measure of output should include imputed rental value of land consumed by the households, i.e. $\tilde{Y}_t = Y_t + \left(\frac{h_t'}{c_t'}\right)^{-\sigma_h} h_t'$. But the results are essentially the same with the current output measure $Y_t$.

$^{19}$Another way to see the nonlinearity is to look at $\frac{dx}{d\omega_t}$, $x = q$ or $Y$, as functions of $\omega_t$, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 3 suggests that this function is decreasing in $\omega_t$ and is much higher at lower $\omega_t$. 

Figure 3: Policy and pricing functions for incomplete markets
ative shock decreases the net worth of the levered entrepreneurs. Due to risk-aversion, these entrepreneurs try to smooth consumption but their debt is not state-contingent so they have to liquidate some of their land holding to maintain a certain level of consumption.\textsuperscript{20,21} Their selling activities depress the price of land, and further lowering their net worth, setting off the vicious circle of falling land price and falling net worth. At the heart of this circle is the pecuniary externality a la Geanakoplos and Polemarchakis (1986) due to market-incompleteness. i.e., when selling off their land holding, each entrepreneur does not take into account the negative effect of falling land price on the net worth of other entrepreneurs. To illustrate this point, Figure 4 plots the portfolio choice (land and bond holdings) of the entrepreneurs as function of wealth distribution given the current state is normal (solid blue lines). The figure also plots the portfolio choice next period if the economy stays in the normal state (dashed purple lines) or if the economy enters a recession (dotted red lines). The figure shows that after a bad shock, the entrepreneurs reduce their land holding, as well as borrowing.\textsuperscript{22}

Quantitatively, Row 2 in Table 1 shows the average (over the stationary distribution) of changes in land price and output given the current normal state, $s_t = N$: $\frac{x_{t+1} - x_t}{x_t}$, where $x = q$ or $Y$. We observe significant amplification and asymmetric effects under incomplete markets, even though these effects are smaller compared to the responses in the model with collateral constraint below. The last row of Table 1 shows the the changes of land prices to shocks in the long run of complete markets equilibrium, presented in Appendix A. Compared to the complete markets outcomes, the model with incomplete markets exhibits both amplification and asymmetric effects. Lastly, Table 1 only shows the average responses, due to the nonlinearity of the solution shown in Figure 3, the amplification and asymmetric effects are also much larger conditional on the lower values of $\omega_t$.

\textsuperscript{20}The lower left panel of Figure 3 shows that, unlike land price or output, the consumption of the entrepreneurs only changes linearly with their wealth even when their wealth is very low.

\textsuperscript{21}This corresponds to the fire-sale phenomenon described in Shleifer and Vishny (1997) because, the entrepreneurs (the specialists) are the only agents in the economy who can use land to produce output, as the result, they are the only natural buyers. The households can only consume land and their marginal utility from land consumption is decreasing. So when all entrepreneurs sell a part of their land holding, land price falls significantly.

\textsuperscript{22}The entrepreneurs can also smooth consumption by borrowing more from the households, but similarly they do not take into account the effect of their increased borrowing in increasing interest rate for other entrepreneurs. That is, there is also pecuniary externality in interest rate beside land price. Indeed, interest rates increase so much that the entrepreneurs actually reduce their borrowing, as shown in Figure 4.
Figure 4: Portfolio choice for incomplete markets
3 Incomplete markets with collateral constraint

While the model with incomplete markets deliver amplification and asymmetric responses of the economy to exogenous shocks, adding collateral constraint as in Kiyotaki and Moore (1997) will a priori exacerbate these responses. To quantitatively examine the significance of this constraint in addition to incomplete markets channel presented in the last section, we impose a collateral constraint on the borrowing decision of the entrepreneurs. We use the same global solution method presented in Subsection 2.3.2 to solve for the dynamic stochastic general equilibrium in this model. Besides, with the collateral constraint, the model has a steady state, so we can log-linearize around the steady state (assuming the collateral constraint is always binding) as in Iacoviello (2005). We can then compare the accuracy of the two solution methods.

As in Kiyotaki and Moore (1997) and Iacoviello (2005), we assume a limit on the obligations of the entrepreneurs. Suppose that, if borrowers repudiate their debt obligations, the lenders can repossess the borrower’s assets by paying a proportional transaction cost $(1 - m)E_t [q_{t+1}] h_t$. In this case the maximum amount that a creditor can borrow is bounded by $mE_t [q_{t+1}] h_t$, i.e.

$$b_t + mE_t [q_{t+1}] h_t \geq 0,$$  \hspace{1cm} (13)

where $b_t$ is the saving ($-b_t$ is borrowing) of the entrepreneurs.

Let $\mu_t$ denote the Lagrangian multiplier for entrepreneur’s collateral constraint. The F.O.C for entrepreneurs with respect to land holding is

$$(\pi_t - q_t)c_t^{-\sigma_1} + \mu_t mE_t [q_{t+1}] + \gamma E_t \left[ q_{t+1}c_{t+1}^{-\sigma_1} \right] = 0$$ \hspace{1cm} (14)

and the complementary-slackness condition is

$$\mu_t (b_t + mh_tE_t [q_{t+1}]) = 0.$$ \hspace{1cm} (15)

The F.O.C for the entrepreneurs with respect to bond holding is

$$-p_t c_t^{-\sigma_1} + \mu_t + \gamma E_t \left[ e_{t+1}^{-\sigma_1} \right] = 0.$$ \hspace{1cm} (16)

We rewrite (14) as

$$q_t = \pi_t + \gamma E_t \left[ q_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma_1} + \mu_t mE_t [q_{t+1}] c_t^{\sigma_1} \right].$$
The first two terms on the right hand side of this equation show that, from the entrepreneurs point of view, land price is the net present value of present and future profit from production using land. In addition, the last term in the right hand side shows the collateral value of land in the land valuation of the entrepreneurs. Iterate this equation forward, we obtain the expression for land price as the present discounted value of profit, with the discount factor depends on the marginal utility of the entrepreneurs as well as on the multiplier on the collateral constraint:

\[
q_t = \pi_t + \mathbb{E}_t \left[ q_{t+1} + \frac{c_{t+1}}{c_t} \right]^{\sigma_1} + \mu_t m c_t^{\sigma_1}
\]

\[
= \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \gamma s {\prod_{r=0}^{s-1} \left( \frac{c_{t+r+1}}{c_{t+r}} \right)}^{\sigma_1} + \mu_{t+r} m c_{t+r}^{\sigma_1} \right] \pi_{t+s}.
\]

Other conditions are the same as in the benchmark incomplete markets model in Section 2. Similar to Lemma 1 in Section 2, we can easily show that \( \omega_t \geq 0 \) in any competitive equilibrium under collateral constraint.

### 3.1 Solution

We can apply the nonlinear global solution method as before, but here a steady state exists, we can also log-linearize around the steady-state and examine the accuracy of the log-linearize solution.

#### 3.1.1 Steady state

Becker (1980) shows that in a neoclassical growth model with heterogeneous discount factors, long run wealth concentrates on the most patient agents, in this case the households. However, in our model, due to collateral constraint, the entrepreneurs can only pledge a fraction of their future wealth to borrow and consume in present time. Therefore, despite their lower discount factor, their wealth does not disappear in the long run. In particular, the model admits a long run steady state in the absence of uncertainty. In this subsection, we solve for the steady state in our model.

Suppose that there is no uncertainty, i.e., \( A_t(s_t) \equiv A \). In steady state, all variables are constant, so we can omit the subscript \( t \). For the ease of notation, denote \( \gamma_e = m \beta + (1 - m) \gamma \). The first order condition for \( b' \), equation (11), implies that \( p = \beta \). Because \( \gamma < \beta \), the

---

\[23\] This formula is similar to the one in Mendoza (2010). In particular when the entrepreneurs cannot borrow against their land holding, i.e., \( m = 0 \), there will not be any collateral premium in the pricing of land.
entrepreneur wants to borrow as much as possible up to the collateral constraint. Indeed, the first order condition for $b$ implies that the collateral constraint is strictly binding and Lagrangian multiplier $\mu$ on the constraint is strictly positive:

$$\mu = (\beta - \gamma) e^{-\sigma_1} > 0.$$ 

Given that the collateral constraint is binding, we have $b = -mqh$.

From the first-order condition (14), we have

$$q = \frac{1}{1-\gamma_e} v Ah^{\nu-1} L^{1-\nu}.$$ 

The steady state version of equation (7) is

$$w = (1 - \nu) Ah^\nu L^{-\nu}. \quad (17)$$

From the budget constraint of the entrepreneurs, we obtain

$$c = \frac{(1 - \gamma)(1 - m)\nu}{1 - \gamma_e} Ah^\nu L^{1-\nu}.$$ 

Combining with the market clearing condition in the market for consumption good, we have $c' = Ah^\nu L^{1-\nu} - c$. The market clearing conditions in the housing market and labor market imply, $h' = H - h$ and $L' = L$.

So in the steady-state all variables can be expressed as functions of two unknowns, $h$ and $L$. The first-order conditions on $h'$ and $L'$ of the households provide two equations that help determine the two unknowns:

$$-q \left(c'\right)^{-\sigma_2} + j \left(h'\right)^{-\sigma_h} + \beta q \left(c'\right)^{-\sigma_2} = 0$$

and

$$w \left(c'\right)^{-\sigma_2} = (L')^{\eta - 1}.$$ 

For example, when $\sigma_2 = 1$ and $\sigma_h = 1$ as in Iacoviello (2005), the second equation, combined with the labor choice equation at the steady state (17) implies

$$L = \left[ \frac{1 - \nu}{(1-\gamma)(1-m)\nu} \right]^{\frac{1}{\eta}}.$$
From the first equation, $h$ is determined as

$$
\frac{h}{H} = \frac{v (1 - \beta)}{v (1 - \beta) + \beta \frac{1}{1 - \gamma} \gamma (1 - m) v}.
$$

Given the determination of the steady state level of $h$ and $L$, the steady state level of wealth distribution defined in (6) is

$$
\omega = \frac{(1-m)h}{H}.
$$

### 3.1.2 Log-linearization

Following Iacoviello (2005), we assume that the collateral constraint always binds around the steady state. Relative to the standard log-linearization technique, we need to solve for the shadow value of the collateral constraint, i.e., the multiplier $\mu_t$, in addition to prices and allocation. Given a variable $x_t$, let $\hat{x}_t$ denote the percentage deviation of $x_t$ from its steady state value, i.e.,

$$
\hat{x}_t = \frac{x_t - x}{x}.
$$

Given the exogenous processes for technology shock $\hat{A}_t$, we solve for the endogenous variables $\hat{c}_t, \hat{c}_t', \hat{h}_t, \hat{h}_t', \hat{b}_t, \hat{q}_t, \hat{w}_t, \hat{p}_t, \hat{L}_t, \hat{\mu}_t$ using the method of undetermined coefficients.\(^{24}\)

The following linear system characterizes the dynamics of the economy around the steady state:

$$
(\hat{q}_t - \sigma_2 \hat{c}_t') = -\sigma_2 (1 - \beta) \hat{H}_t + \beta \hat{E}_t [(\hat{q}_{t+1} - \sigma_2 \hat{c}_{t+1}')] \\
\hat{p}_t = \sigma_2 (\hat{c}_t' - \hat{E}_t \hat{c}_{t+1}') \\
\hat{w}_t - \sigma_2 \hat{c}_t' = (\eta - 1) \hat{L}_t \\
(1 - \gamma_e) [\hat{A}_t + (v - 1) \hat{h}_t + (1 - v) \hat{L}_t - \sigma_1 \hat{c}_t] - (\hat{q}_t - \sigma_1 \hat{c}_t) \\
+ m (\beta - \gamma) (\hat{\mu}_t + \hat{E}_t \hat{q}_{t+1}) + \gamma \hat{E}_t (\hat{q}_{t+1} - \sigma_1 \hat{c}_{t+1}) \\
= 0
$$

$$
\beta (\hat{p}_t - \sigma_1 \hat{c}_t) = (\beta - \gamma) \hat{\mu}_t - \gamma \sigma_1 \hat{E}_t (\hat{c}_{t+1}) \\
\hat{w}_t = \hat{A}_t + v \hat{h}_t - v \hat{L}_t \\
\hat{b}_t' = \hat{h}_t + \hat{E}_t \hat{q}_{t+1} \\
c^* \hat{c}_t + c^* \hat{c}_t' = Y^* [\hat{A}_t + v \hat{h}_t + (1 - v) \hat{L}_t] \\
h^* \hat{h}_t + h^* \hat{h}_t = 0
$$

\(^{24}\)Given the special 3-state structure of the stochastic shocks assumed in Subsection 2.4, we cannot directly use Dynare to solve for the log-linearization version of the model.
\[ c' \hat{c}' + q h' (\hat{h}'_t - \hat{h}'_{t-1}) + \beta b'^\alpha (\hat{p}_t + \hat{b}') = b'^\alpha \hat{b}'_{t-1} + w L (\hat{\omega}_t + \hat{L}_t). \]

### 3.2 Numerical results

In this subsection, we report the properties of the numerical solution of our benchmark model with the parameters given above, in particular the size of the productivity shock is chosen at 3%. Moreover, we set the margin \( m = 0.89 \) as in Iacoviello (2005). The most important properties are the following.

First of all, the fully nonlinear solution for Markov equilibrium features occasionally binding collateral constraint. Collateral constraint (13) binds when the entrepreneurs’ wealth is sufficiently low. In this binding region, endogenous variables including land price and output are more sensitive to changes in wealth distribution. Second, the equilibrium is asymmetric with respect to bad shocks versus good shocks despite the fact that the stochastic structure of the shocks is totally symmetric. For example, on average, good shocks increase land price less than bad shocks decrease land price. Third, the log-linearization solution, by assuming always binding collateral constraint, overestimates the effect of the shocks. Lastly, the probability of binding collateral constraint decreases rapidly with the size of the shocks due to precautionary saving motive of the entrepreneurs. Indeed, in the stationary distribution, the binding probability is 83.76% for 1% standard deviation of the shocks and 5.43% for 5% standard deviation of the shocks.

Figure 5 shows the equilibrium price, output, and consumption of the consumers and entrepreneurs for 3% shocks as function of \( \omega \) in recession (thick blue lines) compared against the same functions under incomplete markets only in the benchmark model (thindashed red lines). Under collateral constraint, the functions exhibit significantly more nonlinearity when \( \omega_t \) is close to zero. This is because the collateral constraint is binding when \( \omega_t \) is close to 0. When the collateral constraint binds, the standard feedback effect in Kiyotaki and Moore (1997) kicks in: after a negative productivity shock, even temporary, in order to smooth consumption the entrepreneurs have to cut back their land holding, \( h_t \). This reduction in land demand depresses land price, \( q_t \), which through the collateral constraint, forces the entrepreneurs to reduce their debt, partly by reducing consumption, and further cut back their land holding. This vicious circle results in significant decline in land price, as well as, in entrepreneurs’ land holding and total output. There is also an intertemporal feedback process, lower current wealth of the entrepreneurs leads to lower future wealth and lower future land prices. Given that the current land price is a weighted value of current per unit profit and future land price, as shown in the asset
Figure 5: Policy functions for collateral constraint versus incomplete markets, $s_t = B$.

pricing equations, lower future land price in turns leads to lower current land price. The entrepreneurs use land to produce so a significant decrease in land holding leads to a significant decrease in output.

The advantage of the nonlinear solution method used in this paper is that, we can see clearly two regions of the state space in which the economy follow different dynamics. In one region, when the collateral constraint is binding (or nearly binding), the feed-back effect is important. In the other region when the collateral constraint is far from binding, asset price and output are less sensitive to changes in wealth distribution.

Figure 6 corresponds to Figure 2 for this case with collateral constraint. Definition 2 of Markov equilibrium and the algorithm in Subsection 2.3.2 provides the evolution of wealth distribution over time. To illustrate the method, the left panel of Figure 6 shows $\omega_{t+1}$ as functions of $\omega_t$ when the current state $s_t = N$. Next period wealth distribution depends on the realization of the exogenous state $s_{t+1}$. By assumption on the stochastic structure of shocks described in Subsection 2.4, $s_{t+1}$ can be $G$, $N$ or $B$. As shown in

\[ 25 \]

As noticed in footnote 19, we can see different dynamics between the two regions is to look at $\frac{dx_t}{d\omega_t}$, $x = q, Y$, as functions of $\omega_t$, i.e. the marginal effect of redistributing wealth from the households to the entrepreneurs to the households on land price and output. Figure 5 shows that this function is decreasing in $\omega_t$ and is much higher at lower $\omega_t$ when the collateral constraint is binding.
the figure, when $\omega_t$ close to $\omega_j$, we have $\omega_{t+1} > \omega_t$, thus the lowest levels of wealth is never reached. This result is in contrast to the one in the benchmark model (or in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013)). Using these transition functions for wealth distribution, we can compute the stationary distribution for wealth distribution, $\omega_t$, over the business cycles. The right panel of Figure 6 shows the stationary distribution (density) for $\omega_t$ conditional on the exogenous state $s_t$. The density becomes zero before the lowest thresholds $\omega_j$.

The most importance difference between the model with collateral constraint and the benchmark incomplete markets model is the possibility of binding collateral constraint. Figure 7 illustrates this point. The upper panel shows the portfolio choice of the entrepreneurs as function of $\omega_t$ and $s_t = N$. On the right of the vertical green line (in both panels), the collateral constraint is binding. This yields the (conditional) binding probability in normal times of 23.49%. The binding probabilities for other states are given in Table 3. Because of the important nonlinearity when the collateral constraint is binding, dynamically the entrepreneurs try to avoid this region by precautionary saving. Precautionary saving decreases the likelihood of binding constraint, and significantly so when
Figure 7: The entrepreneurs’ portfolio choice and the stationary distribution of wealth in normal times shocks are large.

Figure 8 compares the impulse-response of the log-linearization versus the global nonlinear method. Starting from the long run mean level of wealth, we assume the economy is hit by a sequence of 4 good shocks (dotted red line) and 4 bad shocks (solid blue line) respectively and returns to the normal state afterwards.\(^{26,27}\) In the case of bad shocks, we plot the minus of relative changes in land price. This figure illustrates the asymmetric effect of collateral constraint. Positive shocks increase the land price by only 3.2% at impact, while negative shocks decrease land price by 3.8% at impact. Negative shocks also have more persistent effect on land price. The black dotted line shows the IRF under log-linearization. Under log-linearization, responses to shocks are perfectly symmetric so we only plot the IRF under positive shocks. As shown in the figure, by assuming that the

\(^{26}\)As documented in Subsection 2.4, the average length of recessions is 3.6 quarters, so we use 4 shocks for the impulse responses.

\(^{27}\)Another way to plot the impulse-response is to simulate the economy using the transition matrix \(\Pi\) in Subsection 2.4 starting from a good shock or a bad shock, and take the average dynamics of the economy across simulations.
collateral constraint is always binding, log-linearization overstates the effects of shocks. Land price changes by close to 5% at impact and the changes are also more persistent.

Another way to capture the asymmetric effect of collateral constraint is to calculate the average (weighted by the stationary distribution) percentage change in land price and output in normal times, i.e., $s_t = N$, when the shocks hit the economy. The third row of Table 1 shows that good shock changes price by 3.30% and output by 3.15% on average while bad shock changes price by 3.76% and output by $-3.5\%$. Compared to the complete markets outcomes (last row), the collateral constraint exhibit both amplification and asymmetric effects.\footnote{Even though the amplification effect here is relatively small - land price declines by 3.76\% after bad shock, on average, compared to 3\% under complete markets - the effect will be significantly larger if we increase the share of land, $\nu$, in the production function as in Kocherlakota (2000). Moreover, land price also declines much more after bad shocks when the collateral constraint is binding.}

The difference between the log-linearization solution and fully nonlinear solution depicted in Figure 8 comes from the assumption that the collateral constraint always binds around the steady state. This assumption becomes less accurate since as the size of shock increases because agents would engage more in precaution saving. As a result the collateral constraint will not bind all the time. Table 3 shows the binding probability in the long run stationary distribution for wealth distributions as function of the size of the shock.

Figure 8: IRF for non-linear versus log-linear model
Table 3: probabilities of binding constraint

<table>
<thead>
<tr>
<th>△A</th>
<th>Expansion</th>
<th>Normal</th>
<th>Recession</th>
<th>Overall</th>
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</thead>
<tbody>
<tr>
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<td>98.04%</td>
<td>99.83%</td>
<td>83.76%</td>
</tr>
<tr>
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<td>86.06%</td>
<td>36.64%</td>
</tr>
<tr>
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<td>0%</td>
<td>23.49%</td>
<td>70.22%</td>
<td>27.14%</td>
</tr>
<tr>
<td>4%</td>
<td>0%</td>
<td>4.89%</td>
<td>40.40%</td>
<td>9.70%</td>
</tr>
<tr>
<td>5%</td>
<td>0%</td>
<td>0.53%</td>
<td>32.30%</td>
<td>5.43%</td>
</tr>
</tbody>
</table>

Column 2-4 shows the binding probabilities conditional on the realization of the exogenous shocks, and column 5 shows the unconditional binding probabilities. The binding probabilities decrease very fast in the size of the shock.\(^29\)

We end this subsection by noting that our fully nonlinear solution does not exhibit the volatility paradox presented in Brunnermeier and Sannikov (2014), i.e., lower exogenous risk can lead to higher endogenous risk. As shown in Table 3, as we decrease the size of the exogenous shocks, the binding probability goes to 1 and the nonlinear solution becomes closer to the log-linear solution, and both converge to the steady state with no endogenous risk. The difference between our solution and Brunnermeier and Sannikov (2014)’s comes from the fact that we do not allow the households to start producing when the entrepreneurs’ wealth goes to zero. If we assume, as in Brunnermeier and Sannikov (2014), that the households can use an alternative, inefficient production technology, $Ah^\nu L^{1-\nu}$, where $\min(A_t) < A < \text{mean}(A_t)$ to produce, then we should recover the volatility paradox. In Appendix D, we present a simple model with this feature and show that the solution method in this paper applies for that model as well.

### 3.3 Alternative collateral constraint

In the collateral constraint (13), we use the expected future land price. This constraint can be micro-founded under limited commitment and has been used in a large number of papers including Kiyotaki and Moore (1997), Iacoviello (2005), and Cao (2011).\(^30\) However, In Appendix C of Iacoviello (2005), the author shows that the binding probability is close to 1 when the size of the shocks is calibrated to the U.S. data. The binding probability is much smaller in this paper because of our different stochastic structure of the productivity shocks. For example, our stochastic structure implies more persistent shocks (which leads to more precautionary saving) given the same standard deviation of the shocks.

\(^30\)Cao (2011) shows that when the lender can seize a fraction of the asset upon default, the collateral constraint arises endogenously and has the form

$$b_t + mh_t \min_{s_{t+1} | s_t} q_{t+1} (s_{t+1}^{r+1}) \geq 0.$$
in practice collateralized contracts are often written using current asset prices and this is also assumed in a large number of papers, for example Mendoza (2010). Figure 6 shows that wealth distribution moves very slowly over the business cycles, as a result land price also moves slowly. Therefore using current or expected future land price should not imply quantitatively significant differences between the two model. We can show this result rigorously by solving an alternative model in which the collateral constraint (13) is replaced by the following alternative collateral constraint:

\[ b_t + mq_t h_t \geq 0. \]

Fortunately this case is a special case of the general Markov equilibrium definition and solution method in Kubler and Schmedders (2003) and Cao (2010). We solve for the Markov equilibrium under this alternative collateral constraint for the parameters in Subsection 2.4. The solution is quantitatively similar to the one in our benchmark model. For example, Row 4 in Table 1, shows that the amplification and asymmetric effects are only slightly higher than the ones in the benchmark model.\(^{31}\)

4 Complete markets with collateral constraint

The comparison between the benchmark incomplete markets model in Section 2 and the collateral constraint model Section 3 suggests that one of the main ingredients for the amplification and asymmetric effects is market incompleteness beside the collateral constraint. To further demonstrate this point, we study a variation of the collateral constraint model, which we call the collateral constraint with complete markets model. In this model, we maintain the collateral constraint, however we allow the agents to trade a complete set of Arrow securities, subject to the collateral constraint. In history \( s_t \), let \( p_t(s_{t+1}) \) denote the price of the Arrow security that pays off one unit of consumption good if \( s_{t+1} \) happens and nothing otherwise. Let \( \phi_t(s_{t+1}) \) and \( \phi_t'(s_{t+1}) \) denote the holdings of the entrepreneurs and the households, respectively, of these securities. The definition of competitive equilibrium as well as Markov equilibrium are exactly the same as in the benchmark model, except now we need to impose the condition that the markets for the Arrow securities clear, i.e. \( \phi_t(s_{t+1}) + \phi_t'(s_{t+1}) = 0. \)

\(^{31}\)However, the binding probability is significantly higher at 46.54% compared to 27.14% in the benchmark model.
In this model, the budget constraint of the entrepreneurs become:

\[
c_t + q_t (h_t - h_{t-1}) + \sum_{s_{t+1}|s^t} p_{t} (s_{t+1}) \phi_{t} (s_{t+1}) \leq \phi_{t-1} (s_t) + Y_t - w_t L_t. \tag{18}
\]

and the collateral constraint (13) is now

\[
\phi_{t} (s_{t+1}) + mq_{t+1} h_t \geq 0 \forall s_{t+1}|s^t. \tag{19}
\]

Let \( \tilde{p}_{t} (s_{t+1}) = \frac{p_{t} (s_{t+1})}{\Pr (s_{t+1}|s_t)} \) and \( \mu_{t} (s_{t+1}) \Pr (s_{t+1}|s_t) \) denote the multiplier on the constraint (19) for each \( s_{t+1}|s^t \). The first-order condition on \( \phi_{t} (s_{t+1}) \) implies

\[
- \tilde{p}_{t} (s_{t+1}) c_t^{-\sigma_1} + \mu_{t} (s_{t+1}) + \gamma (c_{t+1} (s^t, s_{t+1}))^{-\sigma_1} = 0 \tag{20}
\]

and the first-order condition on the \( h_t \) implies

\[
(\pi_t - q_t) c_t^{-\sigma_1} + m\mathbb{E}_t [\mu_{t} (s_{t+1}) q_{t+1}] + \gamma \mathbb{E}_t [q_{t+1} c_{t+1}^{-\sigma_1}] = 0. \tag{21}
\]

Similarly, the budget constraint of the households changes to:

\[
c_t' + q_t (h_t' - h_{t-1}') + \sum_{s_{t+1}|s^t} p_{t} (s_{t+1}) \phi_{t} (s_{t+1}) \leq \phi_{t-1} (s_t) + w_t L_t'.
\]

From the optimal decision of the households, we have

\[
- \tilde{p}_{t} (s_{t+1}) c_t'^{-\sigma_2} + \beta (c_{t+1} (s^t, s_{t+1}))^{-\sigma_2} = 0. \tag{22}
\]

Other conditions, including the first-order condition with respect to \( h_t' \) of the households stay the same as in the collateral constraint with incomplete markets model.

Figure 9 shows the differences between the price and policy functions in collateral constraint with complete markets model and collateral constraint with incomplete markets model (when \( s_t = B \)). In contrast to the incomplete markets model, land price in collateral constraint with complete markets model differs from land price in the collateral constraint with incomplete markets model for intermediate levels of wealth of the entrepreneurs.

More importantly, numerical simulations show that, unlike the cases with incomplete markets (with or without collateral constraint or ), the long run stationary distribution of wealth is degenerate and concentrates on \( \omega^* \), regardless of the exogenous state. At \( \omega_t = \omega^* \), collateral constraint is binding for all future states, and land demand of the
Figure 9: Price and policy functions under complete insurance versus collateral constraint markets, $s_t = B$. 
entrepreneurs is given by $h^*$. Therefore: $\omega^* = \frac{q_{t+1}h^* - mq_{t+1}h^*}{q_{t+1}H} = (1 - m) \frac{h^*}{H}$. In Appendix C, we present other equations that determine the equilibrium at $\omega^*$. The parameters in Subsection 2.4 implies $\omega^*$ around 0.0235.

At $\omega^*$, Table 1 shows that both the amplification and asymmetric effects disappear. In order to understand how the complete set of Arrow securities help the entrepreneurs to insure against negative shocks. Figure 10, lower panel, shows the entrepreneurs’ optimal choice of $\phi_t(s_{t+1})$ as function of the wealth distribution and the future exogenous state $s_{t+1}$, given the current state $s_t = N$. For clarity we also plot the choice of $\phi_t$ for $s_{t+1} = G$ and $s_{t+1} = B$. Below $\omega^*$, the collateral constraints are binding for all future states. However, above but close to $\omega^*$, the collateral constraints are not binding, and the entrepreneurs borrow relatively more from the future good state compared to the future bad state.

The solution of this model with collateral constraint but complete markets shows that when the agents have access to a complete set of securities, amplification and asymmetric effects disappear. This point is also made by Krishnamurthy (2003) in a simple 3-period economy and DiTella (2014) in a continuous-time model.\footnote{However, DiTella (2014) does not impose any collateral constraint but instead assumes some moral hazard problems that constrains the entrepreneurs’ borrowing capacity.}

Figure 10: Land and arrow securities holdings, $s_t = N$
5 Conclusion

In this paper, we have shown that market incompleteness, independently of the collateral constraint, plays a quantitatively significant role in the amplified and asymmetric responses of the economy to exogenous shocks. There is only one type of shocks - productivity shocks. However, it is easy to extend the paper to incorporate other shocks such as housing preference shocks. It would also be interesting to incorporate money into the model to consider the effect of monetary shocks as in Iacoviello (2005).

The current model does not have capital. We can consider adding capital into the model to examine how the amplification and asymmetric effects affect the capital accumulation and the aggregate production processes. Cao (2010) offers a way to introduce capital into this kind of model which enable the use of a similar global nonlinear solution method to the one in Subsection 2.3.2. However, in this case we need to keep track of two endogenous state variables: wealth distribution and aggregate capital.

The importance of market incompleteness shown in this paper also suggests that state-contingent debt can be an important macro-prudential policy tool. Using a model with collateral constraint, Geanakoplos (2010) argues that leverage should be restricted in booms to avoid fire-sale externality and financial crises in subsequent recessions. Given that market incompleteness plays an important role, designing debts with some insurance for downturn should also be effective in reducing the magnitude of the subsequent recessions. Theoretically, Section 4 shows that complete state-contingent assets, even being subject to collateral constraints, can nullify the amplification and asymmetric effects. In practice, for example, Mian and Sufi (2014) argue that share-responsible mortgages, i.e., mortgages that reduce principal and mortgage payments upon significant declines in housing prices, can significantly reduce the size of the financial and economic crisis 2007-2008 in the U.S.
Appendix

A Complete markets

When markets are complete, the entrepreneurs are less patient than the households, so they tend to consume all their future net worth at the present. When they are risk-neutral as in Brunnermeier and Sannikov (2014), they will consume their entire net worth at time 0, but here they are risk-averse, so they consume their net worth overtime, and their wealth relative to the households’ wealth goes to zero as time goes to infinity. The entrepreneurs can do so by issuing equity to households thanks to frictionless financial markets. We consider the long run limit, when the wealth of the entrepreneurs is close to zero. The economy converges to a representative household economy, with the production technology of the entrepreneurs. Due to the Markovian feature of uncertainty, the endogenous variables depend only on the aggregate state \( s_t: x_t = x(s_t) \).

Given that the households own the whole production sector, the marginal utility from the marginal profit per unit of land should be equal to the marginal utility of one unit of land consumption, i.e., \( (c'_t)^{-\sigma_2} \pi_t = j(h'_t)^{-\sigma_h} \). To evaluate the marginal utility of consumption, we observe that the households consume the whole output in the long run so

\[
c'_t = Y_t = A_th^v_t L^{1-v}_t.
\]

From the equalization of the marginal utility from the marginal profit and marginal utility from land consumption, and using the expression for wage and profit in Subsection 2.3, we have

\[
j (H - h_t)^{-\sigma_h} = \left( A_th^v_t L^{1-v}_t \right)^{-\sigma_2} v A_t h^v_t L^{1-v}_t.
\]

The consumption and labor trade-off equation (12) implies

\[
(1 - v) A_t h^v_t L^{-v}_t \left( A_t h^v_t L^{1-v}_t \right)^{-\sigma_2} = L^v_t - 1.
\]

The two equations (23) and (24) help us solve for the two unknowns \((h_t, L_t)\) for each \(A_t\).

Now, given \(\pi_t\) as function of \(A_t, h_t,\) and \(L_t\), land price is determined by the pricing kernel using the marginal utility of the representative households:

\[
q(s_t) - \pi(s_t) = \sum_{s_{t+1}|s_t} \beta u'(c_{t+1}(s_{t+1})) \frac{u'(c_t(s_t))}{u'(c_t(s_t))} q(s_{t+1}) Pr(s_t, s_{t+1}).
\]
B Incomplete markets with exogenous borrowing constraint

In this Appendix, we examine an alternative model with exogenous borrowing constraint. The model has the same ingredients as the one in Section 2 except for the following borrowing constraint instead of the collateral constraint (13):

\[ b_t \geq -\bar{B}. \]  

(25)

The borrowing constraint \( \bar{B} \) is chosen at the steady state level of debt of the original model. Let \( \mu_t \) denote the Lagrangian multiplier associated to this borrowing constraint. The first-order condition with respect to \( h_t \) and \( b_t \) in the maximization problem of the entrepreneurs implies

\[ (\pi_t - q_t)c_t^{-\sigma_1} + \gamma \mathbb{E}_t[q_{t+1}c_t^{-\sigma_1}] = 0 \]  

(26)

and

\[ -p_t c_t^{-\sigma_1} + \mu_t + \gamma \mathbb{E}_t \left[ c_{t+1}^{-\sigma_1} \right] = 0, \]  

(27)

and the complementary-slackness condition is satisfied:

\[ \mu_t (b_t + \bar{B}) = 0. \]  

(28)

Other conditions are the same as in the model with endogenous borrowing constraint in Section 2.

We first solve for the steady state of this model. As in Section 2, from the first-order condition of the households, we have \( p = \beta \). From the first-order condition (27), we have

\[ \mu_t = (\beta - \gamma) c_t^{-\sigma_1}. \]

From the first-order condition (26), we have

\[ q = \frac{1}{1 - \gamma} \nu Ah^{\nu-1}L^{1-\nu}. \]

This expression of price is different from the one in Section 2 in the discount factor \( \gamma \) instead of \( \gamma^e \). Given that \( \gamma < \gamma^e \), given the same steady state level of \( h \) and \( L \), the land price is lower under exogenous borrowing constraint than under endogenous borrowing constraint. Consequently, this model with exogenous borrowing constraint and the collateral constraint model do not share the same steady state.

Outside the steady state, we can use the global nonlinear solution method presented
in Subsection 2.3.2 to solve for Markov equilibrium in this economy. Table 4 is the counter part of Table 1 for this model with exogenous borrowing constraint. In particular, Row 2 of Table 4 corresponds to Row 2 in Table 1, in which there is no upper bound on the borrowing of the entrepreneurs. When we tighten the exogenous constraint, the amplification and asymmetric effects are actually reduced. At first sight, this result seems counter-intuitive. However, this result is in line with the discussions in Mendoza (2010) and Kocherlakota (2000). The exogenous borrowing constraint reduces the borrowing of the entrepreneurs, and thus reduces the net worth effect in the benchmark incomplete markets model. An important difference here compared to Kocherlakota (2000), is that under uncertainty, it is possible to have infinite exogenous borrowing constraint in the incomplete markets model (the entrepreneurs limits themselves from borrowing too much because of the precautionary saving motive). Infinite exogenous borrowing constraint leads to maximal net worth effect, thus significant amplification and asymmetric effects.

### Table 4: Average land price and output changes in normal state, 3% shock

<table>
<thead>
<tr>
<th>Type of constraint</th>
<th>Land price</th>
<th></th>
<th>Output</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Recession</td>
<td>Expansion</td>
<td>Recession</td>
</tr>
<tr>
<td>incomplete markets ($B = \infty$)</td>
<td>3.21%</td>
<td>-3.45%</td>
<td>3.25%</td>
<td>-3.44%</td>
</tr>
<tr>
<td>incomplete markets ($B = 2$)</td>
<td>3.21%</td>
<td>-3.23%</td>
<td>3.15%</td>
<td>-3.18%</td>
</tr>
<tr>
<td>incomplete markets ($B = 1$)</td>
<td>3.14%</td>
<td>-3.13%</td>
<td>3.04%</td>
<td>-3.04%</td>
</tr>
<tr>
<td>complete markets</td>
<td>3.00%</td>
<td>-3.00%</td>
<td>2.97%</td>
<td>-2.97%</td>
</tr>
</tbody>
</table>

In Subsection 2.3.2 to solve for Markov equilibrium in this economy. Table 4 is the counter part of Table 1 for this model with exogenous borrowing constraint. In particular, Row 2 of Table 4 corresponds to Row 2 in Table 1, in which there is no upper bound on the borrowing of the entrepreneurs. When we tighten the exogenous constraint, the amplification and asymmetric effects are actually reduced. At first sight, this result seems counter-intuitive. However, this result is in line with the discussions in Mendoza (2010) and Kocherlakota (2000). The exogenous borrowing constraint reduces the borrowing of the entrepreneurs, and thus reduces the net worth effect in the benchmark incomplete markets model. An important difference here compared to Kocherlakota (2000), is that under uncertainty, it is possible to have infinite exogenous borrowing constraint in the incomplete markets model (the entrepreneurs limits themselves from borrowing too much because of the precautionary saving motive). Infinite exogenous borrowing constraint leads to maximal net worth effect, thus significant amplification and asymmetric effects.

### C Stationary state under collateral constraint with complete markets

We look for an equilibrium in which prices and allocations depend only on the exogenous state $s_t$. In this case we simplify the notation of state contingent prices and bond holdings by $\tilde{p}_t(s_{t+1}) = \tilde{p}(s_t,s_{t+1})$ and $\phi_t(s_{t+1}) = \phi(s_t,s_{t+1})$. Moreover, we look for the equilibrium in which collateral constraints (19) are all binding, i.e., $\varphi^*(s_t,s_{t+1}) = -mq(s_{t+1})h(s_t)$. This implies $\omega_{t+1} = (1 - m) \frac{h_t}{H}$. In order for $\omega_{t+1}$ not to depend on $s_{t+1}$, we must have then $h_t = h^*$ independent of the exogenous state.

From the budget constraint of the entrepreneurs, equation (18), we have

$$c(s_t) = \sum_{s_{t+1} | s_t} p(s_t,s_{t+1}) mq(s_{t+1}) h^*$$

$$- mq(s_t) h^* + A(s_t) (h^*)^\nu (L_t)^{1-\nu} - w(s_t) L_t$$

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and by the market clearing condition for consumption good, we have

\[ c'(s_t) = A(s_t) (h^*)^\nu (L(s_t))^{1-\nu} - c(s_t). \]

Given \( h^* \) and \( L(s_t) \), \( w(s_t) \) is determined by the first-order condition from the entrepreneurs' optimal choice of \( L_t \), i.e., equation (7). Therefore, we only need to solve for

\[ \{h^*, q(s_t), \tilde{p}(s_t,s_{t+1}), L(s_t)\}. \]

The first-order condition with respect to \( \phi'_t(s_{t+1}) \) implies

\[ \tilde{p}(s_t, s_{t+1}) = \beta \left( \frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2}. \]

We can choose \( \mu(s_t, s_{t+1}) \) so that equation (20) is satisfied

\[ \mu(s_t, s_{t+1}) = \tilde{p}(s_t, s_{t+1}) c_t^{-\sigma_1} - \gamma (c_{t+1}(s_{t+1}))^{-\sigma_1} \]
\[ = \beta \left( \frac{c'(s_{t+1})}{c'(s_t)} \right)^{-\sigma_2} c_t^{-\sigma_1} - \gamma (c_{t+1}(s_{t+1}))^{-\sigma_1} \]
\[ > 0. \]

Plugging this expression for \( \mu(s_t, s_{t+1}) \) into equation (21), we obtain another set of equations that help determine \( \{q(s_t)\} \). The equations that determine \( \{L(s_t)\} \) comes from the optimal labor-consumption decision of the households, equation (12). And lastly, \( h^* \) must be determined so that the first-order conditions on the housing choice of the household are satisfied in all exogenous states:

\[ \{q(s_t) - \mathbb{E}_t [q(s_{t+1}) \tilde{p}(s_t, s_{t+1})]\} (c'(s_t))^{-\sigma_2} = j(H - h^*)^{-\sigma_p}, \]

given the expression of \( \tilde{p}(s_t, s_{t+1}) \) derived above.

\section*{D Simpler model}

In this Appendix, we simplify our model in the spirit of Brunnermeier and Sannikov (2014) as well as Cordoba and Ripoll (2004) and Kiyotaki and Moore (1997). We assume that the households does not have a preference for housing but have access to an inefficient production function

\[ Y' = Ah^{\nu'} L^{1-\nu'} \]
with \( \min (A_t) < A < \text{mean} (A_t) \). Households maximize a lifetime utility function given by
\[
E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(c_t')^{1-\sigma_2} - 1}{1-\sigma_2} - \frac{1}{\eta} (\bar{L}_t)^{\eta} \right\},
\]
(30)
where \( \bar{L}_t \) is the hours of work (instead of \( L'_t \) in the benchmark model). The budget constraint of the households is
\[
c_t' + q_t (h'_t - h'_{t-1}) + p_t b'_t \leq b'_{t-1} + w_t \bar{L}_t + Y'_t - w_t L'_t.
\]
(31)
Housing is no longer in the utility function of households, so we have to impose explicitly \( h'_t \geq 0 \).

Given their land holding at time \( t, h_t \), the households choose labor demand \( L'_t \) to maximize profit
\[
\max_{L'_t} \{ Y'_t - w_t L'_t \}
\]
subject to their production technology (29). The first order condition with respect to \( L'_t \) implies
\[
w_t = (1 - \nu') A h'_t L'^{-\nu'},
\]
i.e. \( L'_t = \left( \frac{(1-\nu') A_l}{w_t} \right)^{\frac{1}{\nu'}} h'_t \) and profit
\[
Y'_t - w_t L'_t = \pi'_t h'_t
\]
where \( \pi'_t = \nu' A \left( \frac{(1-\nu') A_l}{w_t} \right)^{1-\nu'} \) is profit per unit of land for the households.

**Definition 3.** A competitive equilibrium is sequences of prices \( \{p_t, q_t, w_t\}_{t=0}^{\infty} \) and allocations \( \{c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t, \bar{L}_t\} \) such that (i) \( \{c'_t, h'_t, b'_t, L'_t, \bar{L}_t\} \) maximize (30) subject to budget constraint (31) and the production technology (29), and \( h'_t \geq 0 \) and \( \{c_t, h_t, b_t, L_t\} \) maximize (4) subject to budget constraint (5), collateral constraint (13), and production technology (3) given \( \{p_t, q_t, w_t\} \) and initial asset holdings \( \{h_{-1}, b_{-1}, h'_{-1}, b'_{-1}\} \); (ii) land, bond, labor, and good markets clear: \( h_t + h'_t = H, b_t + b'_t = 0, L_t + L'_t = \bar{L}_t, c_t + c'_t = Y_t \).

In the steady state, the entrepreneurs own the whole supply of land. Outside the steady state, we can use the definition of Markov equilibrium and the associated solution method as in the benchmark model in Section 2. The main difference between the solution of this model and the benchmark model is that at the natural borrowing limit
for the entrepreneurs, i.e. \( \omega_t = 0 \), the households start producing using their inefficient production function. This puts a lower bound on the total output as well as land price.

References


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