When is the Trend the Cycle?

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Abstract

A growing literature debates the explanations for the cyclical properties of emerging markets based on the RBC small open economy model. Two leading explanations are considered: trend shocks (Aguiar and Gopinath 2007), and financial frictions (Neumeyer and Perri 2004; Garcia-Cicco, Pancrazi, and Uribe 2010). We provide analytical parameter restrictions that, within this class of models, favor the trend shocks explanation. This effort centers the debate on two issues. First, the trend shocks explanation crucially requires the interest rate to be very insensitive to changes in the stock of debt. This raises concerns about robustness, and quantitative results confirm that the trend shocks explanation is fragile even in the original model specification. Second, it is difficult to obtain a realistic and powerful propagation of trend shocks in the RBC framework: Using the parameter restrictions generates a fall of labor supply after a positive trend shock. Lastly, even when the analytical restrictions are satisfied, trend shocks are still not able to compete against other shocks introduced into the RBC framework in terms of variance decomposition.

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1 Introduction

A large literature seeks to explain a set of important and salient features of emerging market business cycles based on a class of baseline RBC open economy models. Different from developed countries, emerging markets tend to exhibit a larger volatility of consumption than output, and a volatile and countercyclical current account. These features of the data represent a challenge to standard, frictionless, models.

There are two strands of this literature. One strand of the literature, following the seminal contribution by Aguiar and Gopinath (2007), asserts that a standard RBC small open economy à la Schmitt-Grohe and Uribe (2003) subject to permanent productivity shocks (called also nonstationary technology shocks or trend shocks) is capable of accounting for the aforementioned features of emerging economies. This insight has proven quite influential. The logic is that, due to the permanent income assumption, permanent shocks induce large movements in consumption and a volatile and countercyclical current account. This point has been extended in work by Boz, Daude, and Durdu (2011), Naoussi and Tripier (2013), among others. A related strand of the literature, initiated by Neumeyer and Perri (2004) and Uribe and Yue (2006), asserts instead that financial frictions are the explanation to this large volatility. More recent work by Garcia-Cicco, Pancrazi, and Uribe (2010), Alvarez-Parra, Brandao-Marques, and Toledo (2013), and Chang and Fernandez (2013), among others, has also provided quantitative models of financial frictions. Most often used are purely exogenous shocks to the interest rate faced by the domestic economy, which are in fact able to generate realistic business cycle dynamics.

Our contribution to this debate is mainly analytical. We prove a proposition that makes explicit the conditions under which the permanent income hypothesis holds exactly in the specification by Aguiar and Gopinath (2007) (henceforth AG). This is important, because as AG emphasize, the permanent income hypothesis forms the basis of an explanation grounded on trend shocks.

Our proposition states that, when the sensitivity of the interest rate to movements in the stock of debt goes to zero and preferences are separable, consumption dynamics are only determined by the long-run of level of productivity (up to a constant):

\[ c_t = \text{constant} \cdot \text{long-run level of productivity}_t \]

Somewhat surprisingly, in this parameter region consumption is completely disconnected from the rest of the model, and, it is highly sensitive to trend shocks. To see this, consider a positive trend shock. Because such a shock raises income at infinity, consumption reacts strongly on impact. Thus, not only consumption tends to be

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1See Durdu (2013) for a recent survey.
very volatile, but the small open economy finances this increase in consumption by borrowing from the rest of the world, i.e., the net exports are countercyclical. So long the interest rate does not increase following this increase in debt, this effect is quantitatively powerful. Instead, if the interest rate increases, this effect is muted through the consumption Euler equation, and the model may not be able to fit the facts.

We discuss two issues raised by the proposition.

First, it clarifies the key economic role of the interest rate sensitivity. It is important that, after an accumulation of debt, the interest rate does not increase by much, or not at all. So long this is true, the effect of trend shocks is quantitatively powerful. Notice that AG (and several other following up papers) indeed fix the sensitivity of the interest rate using a single parameter \( \psi \), calibrated to a very small value of \( \psi = 0.001 \). So far, this strand of the literature has attributed to \( \psi \) just the technical role of delivering stationarity.\(^2\) Our proposition assigns it an economic role, the one of delivering the random-walk permanent income behavior of consumption.

Most importantly, from a quantitative point of view, this analytical condition that the interest rate needs to be very insensitive raises concerns about robustness. We thoroughly study the implications of increasing the value of \( \psi \), while leaving all other parameters used by AG unchanged. We show that when this parameter has a higher value, but still rather small (say 0.1), consumption already features excess smoothness, the volatility of output being higher than the one of consumption. Also, the ratio of the variances of net exports to output goes down to 0.19, whereas in the benchmark results it is 0.71. So, net exports volatility is reduced considerably. In addition, the correlation of net exports and output is also reduced (although to a lesser extent). Thus, assigning a moderate value to \( \psi \) overturns most of the quantitative insights. Using the value \( \psi = 1 \) delivers excess smoothness, a ratio of the variances of net exports to output of 0.10, and correlation of net exports to output more than 2 times smaller, strongly overturning the results in AG. To sum up, it is easy to overturn the permanent shocks explanation by just increasing the sensitivity of the interest rate in the RBC small open economy model.\(^3\)

\(^2\)Our reading of this literature is that, tacitly, it is comfortable with assigning a very low value to \( \psi \) in order to mimic the behavior of the nonstationary model in which the interest rate is simply fixed. Moving away from the nonstationary model to a stationary one provides computational advantages and allows researchers to match second moments (of endogenous model variables in levels) as in the standard RBC closed-economy model. See for instance the discussion in Mendoza (1991) or Schmitt-Grohe and Uribe (2003).

\(^3\)An influential paper by Garcia-Cicco, Pancrazi, and Uribe (2010) (henceforth GPU), appearing after AG, estimates \( \psi \) to 2.8 using Bayesian methods. However, the model used by GPU is not exactly the same as in AG, because it is augmented with interest rate shocks, preference shocks, and spending shocks. To clarify the different findings in AG and GPU, in our numerical exercises, we also consider this estimate (\( \psi = 2.8 \)) in exactly the same model as in AG. It delivers very similar results as using \( \psi = 1 \), overturning the findings in AG. Thus, our proposition underlines the pivotal role played by the behavior of the interest rate.
Second, we analyze the ability of the RBC model in propagating trend shocks realistically. We conclude negatively, for several reasons. First, we notice the following tension. The proposition requires separable preferences. However, this generates a fall in equilibrium labor supply after a positive trend shock, due to the income effect. This feature is closely related to the well-known comovement problem in the news shocks literature (Jaimovich and Rebelo 2009). To resolve this issue, the literature has recurred to GHH preferences (Greenwood, Hercowitz, and Huffman 1988). But, as pointed out by Neumeyer and Perri (2004), this also makes transitory shocks powerful, and likely to compete with permanent shocks as main drivers. To sum up, there is a fundamental tension in using the RBC framework: using preferences that deliver a high volatility of both consumption and net exports disrupt the empirical performance of the model in other aspects.

An immediate reaction here is that one needs more frictions to solve the labor supply issue. However, we point out that a straightforward solution is to resort to an endowment economy. The proof of our proposition does not use the particular specification of the supply structure of the economy usually imposed in the literature. We indeed show that an endowment economy subject to permanent shocks is able to generate the same three key moments emphasized by AG: similar volatility of consumption, similar volatility of the current account, and similar correlation between the current account and output. Of course, the endowment economy is a bit simplistic, but this illustrates that permanent shocks are able to match the three moments emphasized by AG quite generally, so long the sensitivity of the interest rate is very small. At the same time, the relative success of the endowment economy in generating these moments with trend shocks is a strong indication that the mechanism put forth by AG is orthogonal to the spirit of the RBC paradigm.

Lastly, we explore these issues by performing Bayesian estimation. Here, we use the specification and data in GPU, but impose the conditions of our proposition. GPU find that interest rate shocks explain a much higher fraction of business cycle volatility than trend shocks. The question we ask: if one imposes the restrictions of the proposition, do trend perform better in this horse race against other shocks? The answer we obtain strongly favors the propagation mechanism of financial and other shocks within this model. Even when the analytical conditions are imposed, trend shocks explain a very small fraction of both consumption and net exports. Thus, the RBC structure is unable to provide enough traction to trend shocks.

\footnote{This is not a contradiction to our proposition, but only a reflection of which, among financial and trend shocks, are more able to reproduce the dynamics in the data conditional on the cumulative of features of the model. If we remove all non-TFP shocks from the exercise, as expected trend shocks explain virtually all the dynamics of consumption. However, in this case, trend shocks explain very little of the movements of the current account and of output. See Section 4 for details.}
To the best of our knowledge, this is the first paper to take a theoretical approach to analyze the performance of the RBC model in mimicking the economies of emerging markets. In addition, the paper generalizes previous theoretical results (for instance by Campbell and Deaton 1989, or Galí 1991 for endowment economies) by showing that the random-walk permanent income hypothesis for consumption holds exactly, and quite generally, in economies with endogenous investment and production.

The rest of the paper is organized as follows. We setup the model and the log-linear equilibrium in Section 2. We present the proposition and provide an interpretation in Section 3. We explore several quantitative implications of the proposition in Section 4. Section 5 concludes. The proof of our proposition is quite lengthy and it is therefore relegated to the appendix.

2 The Setup and Solution Method

For convenience, it seems natural to use exactly the same model as AG. Thus, we reproduce here the published setup of the model. Then, we reproduce the normalization and log-linearization reported by Aguiar and Gopinath (undated) (henceforth AGb). We use the notation adopted there.

2.1 Aguiar and Gopinath’s 2007 Model

This is a single-good, single-asset, small open economy. Technology is characterized by a Cobb-Douglas production function that uses capital $K_t$ and labor $L_t$ as inputs:

$$Y_t = e^{z_t} K_t^{1-\alpha} (\Gamma_t L_t)^\alpha$$

where $\alpha \in (0,1)$ represents labor’s share of output, and $z_t$ and $\Gamma_t$ are productivity processes. Specifically, level productivity $z_t$ follows

$$z_t = \rho_z z_{t-1} + \epsilon^z_t$$

with $\rho_z < 1$, and $\epsilon^z_t$ is a stationary technology shock, labeled also cycle or transitory shock, and modeled as an i.i.d. draw from a normal distribution with zero mean and standard deviation $\sigma_z$. Trend productivity is modeled with a nonstationary process

$$\Gamma_t = e^{g_t} \Gamma_{t-1}$$

where

$$g_t = \rho_g g_{t-1} + \epsilon^g_t$$
and $\rho_g < 1$. $\epsilon_{t}^{g}$ is a nonstationary technology shock, labeled also trend or permanent shock, and modeled as an i.i.d. draw from a normal distribution with zero mean and standard deviation $\sigma_g$. AG allow for a deterministic trend in $g_t$ (denoted $\mu_g$, see AG p. 80.), but to simplify the algebra in the proof of our main result below, we set this deterministic trend to zero ($\mu_g = 0$). Our result extends easily to the general case with unrestricted $\mu_g$.

Period utility is Cobb-Douglas

$$u_t = \frac{[C_t^{\gamma}(1 - L_t)]^{1-\sigma}}{1 - \sigma}$$

where $0 < \gamma < 1$. The resource constraint is

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t - \frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - 1\right)^2 K_t - B_t + Q_tB_{t+1}$$

where $K_{t+1}$ is capital, $\delta$ is the capital depreciation rate, $B_t$ represents debt due in period $t$, $q_t$ is the time $t$ price of debt due in period $t + 1$ and adjustment costs in capital are captured by

$$\frac{\phi}{2}\left(\frac{K_{t+1}}{K_t} - 1\right)^2 K_t$$

where $\phi > 0$ is a parameter.

The price of debt is sensitive to the level of outstanding debt, taking the form used by Schmitt-Grohe and Uribe (2003):

$$\frac{1}{Q_t} = 1 + r_t = 1 + r^* + \psi \left[\exp\left(\frac{B_{t+1}}{\Gamma_t} - b\right) - 1\right]$$

where $r^*$ is the world interest rate, $b$ represents an exogenous steady-state level of normalized debt, and $\psi > 0$ governs the elasticity of the interest rate to debt.

Normalization and Recursive Formulation. For a variable $X_t$, we write its detrended counterpart by normalizing the variable using previous period’s trend productivity:

$$\hat{X}_t = \frac{X_t}{\Gamma_{t-1}}$$

In normalized form, the representative agent’s problem is written recursively as

$$V(\hat{K}, \hat{B}, z, g) = \max_{\{\hat{C}, \hat{L}, \hat{K}', \hat{B}'\}} \left\{ \frac{[\hat{C}^{\gamma}(1 - L)]^{1-\sigma}}{1 - \sigma} + \beta e^{\sigma \gamma (1-\sigma)} EV(\hat{K}', \hat{B}', z', g') \right\}$$
subject to
\[ \hat{C} + e^g \hat{K}' = \hat{Y} + (1 - \delta) \hat{K} - \frac{\phi}{2} \left( \frac{e^g \hat{K}'}{\hat{K}} - 1 \right)^2 \hat{K} - \hat{B} + e^g q \hat{B}' \]

where a prime on a variable ' denotes the value of the variable at \( t + 1 \).

**Log-linearization.** For nonstationary variables, we define the following log-deviations from stationary steady state quantities:

\[ \hat{c}_t \equiv \log(\frac{C_t}{\Gamma_{t-1}}) - \log(\bar{C}/\bar{\Gamma}) \]
\[ \hat{y}_t \equiv \log(\frac{Y_t}{\Gamma_{t-1}}) - \log(\bar{Y}/\bar{\Gamma}) \]
\[ \hat{x}_t \equiv \log(\frac{X_t}{\Gamma_{t-1}}) - \log(\bar{X}/\bar{\Gamma}) \]
\[ \hat{k}_{t+1} \equiv \log(\frac{K_{t+1}}{\Gamma_t}) - \log(\bar{K}/\bar{\Gamma}) \]

For variables that are already stationary, we define the following log-deviations

\[ \hat{n}_t \equiv \log(\frac{N_t}{\bar{N}}) \]
\[ \hat{l}_t \equiv \log(\frac{L_t}{\bar{L}}) \]
\[ \hat{q}_t \equiv \log(\frac{Q_t}{\bar{Q}}) \]

We then define the absolute deviation of the net exports-to-output ratio

\[ \hat{n} \hat{x}_t = \frac{NX_t}{Y_t} - \frac{\bar{N}X}{\bar{Y}} \]

We also define the absolute deviation of

\[ \hat{b}_{t+1} = \frac{B_{t+1}}{\Gamma_t} - \frac{\bar{B}}{\bar{\Gamma}} \]

These definitions for steady state deviations are identical to the ones used in AGb, with the exception of the last one. There, we use an absolute deviation the relative (log) deviation used in the original paper, in order to allow for \( \bar{B}/\bar{\Gamma} = 0 \). This allows us to obtain general expressions in the proposition below, but our results do not rely on this specification.

The resulting log-linearized model is fully characterized by the following set of equations:
The dynamics of productivity, including the cycle shocks and the trend shocks:

\[ z' = \rho_z z + \epsilon_z \]
\[ g' = \rho_g g + \epsilon_g \]

The first-order condition in \( k' \), which corresponds to equation 12 in AGb:

\[ 0 = (\gamma(1 - \sigma) - 1)E\hat{c}' + (1 - \gamma)(1 - \sigma)E\hat{l}' + \beta\phi E\hat{g}' + \beta(1 - \alpha)\frac{\bar{Y}}{\Gamma}E\hat{y}' + \beta\phi E\hat{k}'' - \left(\beta \left( (1 - \alpha)\frac{\bar{Y}}{\Gamma} + \phi \right) + \phi \right) \hat{k}' - (\gamma(1 - \sigma) - 1)\hat{c} \]
\[ - (1 - \gamma)(1 - \sigma)\hat{l} + (\gamma(1 - \sigma) - 1 - \phi)g + \phi \hat{k} \]  

(AGb12)

The first-order condition in \( b' \), which corresponds to equation 17 in AGb:

\[ 0 = (\gamma(1 - \sigma) - 1)E\hat{c}' + (1 - \gamma)(1 - \sigma)E\hat{l}' + (\gamma(1 - \sigma) - 1)g - (\gamma(1 - \sigma) - 1)\hat{c} - (1 - \gamma)(1 - \sigma)\hat{l} - \hat{q} \]  

(AGb17)

Other equations that describe the dynamics of the log-linearized model are:

\[ 0 = \hat{y} - \hat{n} - \hat{c} + \hat{l} \]
\[ 0 = \frac{\bar{Y}}{\Gamma} \hat{y} + \bar{Q} \hat{b}' + \bar{Q} \frac{\bar{B}}{\Gamma} (g + \hat{q}) - \bar{b} - \frac{\bar{X}}{\Gamma} \hat{x} - \bar{C} \frac{\bar{C}}{\Gamma} \hat{c} \]  

(AGb20)

\[ \frac{\bar{X}}{\Gamma} \hat{x} = \frac{\bar{K}}{\Gamma} \left( \hat{k}' - (1 - \delta)\hat{k} + g \right) \]  

(AGb21)

\[ \hat{y} = z + (1 - \alpha)\hat{k} + \alpha(g + \hat{n}) \]  

(AGb22)

\[ \hat{L} \hat{l} = -\bar{N} \hat{n} \]
\[ \hat{q} = -\psi \bar{Q} \hat{b}' \]  

(AGb24)

\[ \Delta n x = (1 - \bar{N}X/\bar{Y})\hat{y} - \frac{\bar{X}}{\bar{Y}} \hat{x} - \bar{C} \frac{\bar{C}}{\bar{Y}} \hat{c} \]

where we have followed the notation in AGb. The model is exactly the same as in AGb except for (AGb20) and (AGb24). They are different because of the way we normalize the level of debt.

### 3 The Proposition

We consider the case with \( \gamma = 1 \), which corresponds to labor supply being exogenously given. One can trivially restate the arguments for the case \( \gamma \neq 1 \) but \( \sigma = 1 \), which
corresponds to endogenous labor supply, but additively separable from consumption (log-log preferences). Our result holds when preferences belong to one of these two cases. (Numerical simulations also show that the result does not hold outside these two cases.)

If $\gamma = 1$, $n_t = 0$. Under this assumption, the linearization of the production technology (AGb22) becomes

$$\dot{y}_t = z_t + (1 - \alpha) \dot{k}_t + \alpha g_t$$ (2)

We will solve for a log-linearized solution of the system using the state space $X_t = \begin{bmatrix} b_t & \dot{k}_t & \zeta_t & \zeta_{t-1} & z_t \end{bmatrix}'$, where $\zeta_t = \log(\Gamma_t)$ and

$$b_t = \hat{b}_t + \frac{\bar{B}}{\Gamma} \zeta_{t-1}$$

For further use we let $X^0_t = \begin{bmatrix} b_t & \dot{k}_t \end{bmatrix}'$ and $X^1_t = \begin{bmatrix} \zeta_t & \zeta_{t-1} & z_t \end{bmatrix}'$. It is also important to notice that

$$g_t = \zeta_t - \zeta_{t-1}$$

Using the definition of log-consumption, we have

$$c_t = \hat{c}_t + \zeta_{t-1}$$

Following standard log-linearization techniques, for example as presented in Blanchard and Kahn (1980) and Uhlig (1999), the solution to the log-linearized model (AGb12)-(AGb24) takes the form:

$$c_t = D_c X_t$$

$$b_{t+1} = D_b X_t$$

$$\hat{k}_{t+1} = D_k X_t$$ (3)

In particular,

$$c_t = D_c X_t = D^0_c X^0_t + D^1_c X^1_t$$

$$= D_{c,b} b_t + D_{c,k} \dot{k}_t + D_{c,\zeta_1} \zeta_t + D_{c,\zeta_2} \zeta_{t-1} + D_{c,z} z_t$$ (4)

Denote by $\zeta_{t+\infty}$ the expected long-run level of productivity, i.e.

$$\zeta_{t+\infty} = \lim_{j \to \infty} \mathbb{E}[\zeta_{t+j}] = \frac{\zeta_t - \rho g \zeta_{t-1}}{1 - \rho g}$$
We claim that as $\bar{Q} \to 1$ and $\psi \to 0$, consumption is only a function of long-run productivity. Specifically,

$$c_t = \left( \frac{1 - \bar{X}/\bar{Y}}{C/Y} \right) \zeta_{t+\infty}$$  \hspace{1cm} (5)$$

The result is expressed formally as follows.

**Proposition 1**

$$\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,k} = 0$$

$$\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,b} = 0$$

$$\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,\zeta_1} = \left( \frac{1 - \bar{X}/\bar{Y}}{C/Y} \right) \left( \frac{1}{1 - \rho_g} \right)$$

$$\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,\zeta_2} = \left( \frac{1 - \bar{X}/\bar{Y}}{C/Y} \right) \left( \frac{-\rho_g}{1 - \rho_g} \right)$$

$$\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,z} = 0$$

### 3.1 Interpretation

The proposition states that when the interest rate becomes insensitive to changes in debt holdings, consumption is only determined by the long-run level of productivity, as expressed by equation (5). This means, at this limit, the level of debt holdings or the stock of capital do not matter for the determination of consumption. The result also requires that in the steady-state the world interest rate goes to zero ($\bar{Q} \to 1$, following from $\beta \to 1$), which allows the agent to roll-over any existing stock of debt to infinity, thereby allowing to maintain consumption at the long-run level of income (determined by long-run output).\(^5\)

A corollary is that consumption only reacts to permanent shocks, and does not react to transitory shocks. Although not immediately obvious by looking at the proposition, after a permanent shock to productivity consumption jumps to its long-run level $\gamma_{t+\infty}$ and stays there. This can be seen in a probably more transparent way by considering the case of zero steady-state debt holdings (or zero steady-state net exports). The following Corollary considers this case and derives the resulting behavior of $\hat{c}_t$, the normalized log-deviation of consumption.

\(^5\)Campbell and Deaton (1989) and Galí (1991) considered a version of this result in the case of endowment economies. They also focus on the limit of zero interest rate to study the empirical relationship between the variance of consumption changes and of permanent income shocks (transitory income shocks will be negligible in this limit) in a simple setting with constant interest rate and without capital accumulation.
Corollary 1 If $\bar{C}/\bar{Y} + \bar{X}/\bar{Y} = 1$,

$$\lim_{Q \to 1} \lim_{\psi \to 0} \hat{c}_t = \frac{1}{1 - \rho_g} g_t$$

Proof. Proposition 1 shows that in the limit

$$c_t = \left( \frac{1 - \bar{X}/\bar{Y}}{C/Y} \right) \left( \frac{1}{1 - \rho_g} \right) \zeta_t - \left( \frac{1 - \bar{X}/\bar{Y}}{C/Y} \right) \left( \frac{\rho_g}{1 - \rho_g} \right) \zeta_{t-1}$$

from which, given $\bar{C}/\bar{Y} + \bar{X}/\bar{Y} = 1$,

$$c_t = \frac{1}{1 - \rho_g} \zeta_t - \frac{\rho_g}{1 - \rho_g} \zeta_{t-1}$$

Using the definition of $\hat{c}_t$:

$$\hat{c}_t = c_t - \zeta_{t-1} = \frac{1}{1 - \rho_g} \zeta_t - \frac{\rho_g}{1 - \rho_g} \zeta_{t-1} - \zeta_{t-1} = \frac{1}{1 - \rho_g} (\zeta_t - \zeta_{t-1}) = \frac{1}{1 - \rho_g} g_t$$

The constant $(1 - \bar{X}/\bar{Y})/(\bar{C}/\bar{Y})$ in front of equation (5) is simply a factor that adjusts the size of deviations to the value of steady-state variables, which depend on the steady-state capital-to-output and consumption-to-output ratios (both exogenous). When $\bar{C}/\bar{Y} + \bar{X}/\bar{Y} = 1$, this constant is equal to 1. Using this and expressing the result in terms of normalized log-deviations of consumption instead of log-consumption allows to obtain a simple intuitive expression for the behavior of consumption in the AG model, where consumption is equal to the expected cumulated sum of permanent productivity increases $(1/(1 - \rho_g)) \cdot g_t$.

Even though our proposition focuses on consumption, it has indirect implications for the behavior of net exports. The key point is that persistent permanent shocks embed a large wealth effect that generates large short-run volatility on consumption, while they have relatively small effects on output. Thus, the implication is countercyclical and volatile net exports.\(^6\)

\(^6\)

Notice that transitory shocks are sort of a nuisance in the model because they generate extra output volatility (and little consumption volatility), competing with the main channel emphasized here. However, they turn out to be useful to match output volatility.
4 Quantitative Explorations

In this section, we explore the quantitative implications of the proposition. We proceed in four steps. First, we explore the robustness of the ability of the AG model to match three key moments: the relative variance of consumption to output, the relative variance of net exports with respect to output, and the correlation of net exports and output. We look at what happens when the interest rate is more sensitive than the usual calibration in the literature. Second, we look at the implication of using separable preferences. Third, we explore an endowment economy and check its ability to match the three moments just mentioned. Fourth, we perform Bayesian estimation and focus on the variance decomposition of consumption and net exports in a model that includes both trend and financial shocks.

4.1 The Importance of the Sensitivity of the Interest Rate $\psi$

Proposition 1 requires $\psi \rightarrow 0$ in order for consumption to react strongly to changes in the trend of productivity, thereby causing net exports to be highly volatile and countercyclical. We now quantitatively explore this point, and study what happens when $\psi$ is assigned a higher value than the one used in AG (0.001).

All values of the rest of the parameters in this section are the ones used in AG, with the exception of $\psi$ (sensitivity of the interest rate) and $\gamma$, as required by Proposition 1. The other benchmark parameter values are presented in Table 1.

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<table>
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<th>Parameter</th>
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<td>Persistence transitory shock</td>
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<td>Standard dev. transitory shock</td>
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Notes: These parameters for shock processes are the same used for Table 4 of AG, Specification 1, Mexico.

We first check our theoretical result numerically. As shown in Figure 1, imposing
very small (and \( \beta \) close to 1 together with \( \psi/(1 - \beta) \) very small\(^7\)) ensures the random walk behavior of consumption. Consumption jumps on impact following a permanent shock, but does not move at all following a transitory shock. These sharp results are quite striking given the complex structure of the rest of the model.\(^8\)

Figure 1: Impulse Responses: Consumption and Net Exports

![Impulse Responses: Consumption and Net Exports](image)

Notes: The lines depict responses of the model in AG in the limiting case of Proposition 1 where we set \( \beta = 0.99999 \) and \( \psi = 10^{-12} \). We also set \( \gamma = 1 \) which implies that labor supply is exogenously given. The standard deviations of technology shocks are normalized to 1, similar to Figure 3 in AG. The other parameters are from Table 1.

Figure 2 examines the role played by the parameter \( \psi \) in determining the sensitivity of consumption to trend shocks by depicting impulse responses of consumption using different values of \( \psi \). We consider the value used in AG (0.001) along with some larger values (0.01, 2.8). 2.8 is the value estimated by Garcia-Cicco, Pancrazi, and Uribe (2010) (henceforth GPU). Consumption does not immediately reach its long-run level with a permanent shock when the parameter \( \psi \) takes the values we choose here, these values being substantially larger than the one we used previously in Figure 1 (\( \psi = 10^{-12} \)). Notice that, crucially, the larger \( \psi \), the smaller the response of consumption following a permanent shock, and the larger the response following a transitory shock. This tends to overturn the results.

Further examining the role played by the parameter \( \psi \), we reproduce Figure 3 of AG with different values of \( \psi \). Again, we consider \( \psi = 0.001, \psi = 0.01, \) and \( \psi = 2.8 \)

\(^7\)These are equivalent to \( \lim_{Q \to 1} \lim_{\psi \to 0} \) (in that order).

\(^8\)For this Figure, we also set \( \beta = 0.99999 \) and \( \gamma = 1 \) (see p. 10). We have also verified that setting \( \sigma = 1 \) instead delivers exactly the same results for consumption, and similar results for net exports.
and obtain impulse responses of (A) the ratio of net exports to GDP, (B) the ratio of consumption to GDP, and (C) the ratio of investment to GDP following a 1 percent shock on \( \epsilon^g \) and \( \epsilon^z \). We clearly observe that these ratios vary substantially with the parameter \( \psi \). Most importantly, the response of net exports to a permanent \( g \) shock is muted with large values of \( \psi \). The reason is the muted response of consumption. These numerical results reveal a similar behavior of investment, which also features a muted response.

Finally, Table 2 reports a set of moments using different parameter values of \( \psi \), ranging from \( 10^{-12} \) to 2.8, the value estimated by Garcia-Cicco, Pancrazi, and Uribe (2010). We use exactly same parameters used in AG, except for \( \psi \). As shown in the table, the volatility of consumption \( \sigma(c) \), the relative volatility of consumption to output \( \sigma(c)/\sigma(y) \), and that of net exports to output \( \sigma(NX)/\sigma(y) \) depend greatly on the parameter \( \psi \). Also, similar to the results shown in Figure 3, the relative volatility of investment respect to output \( \sigma(I)/\sigma(y) \) is decreasing with \( \psi \).

To sum up, consistent with Proposition 1, this subsection has numerically illustrated the role played by the interest rate sensitivity parameter \( \psi \) for the results in AG. The main point is that a very small value of \( \psi \) ensures the random walk behavior of consumption and leads the model to generate the key moments emphasized in AG. Larger values of \( \psi \) tend to reverse this, the model losing the ability to generate those

Notes: For all specifications, we set \( \beta = 0.98 \) and \( \sigma = 2 \). Also, the standard deviations of technology shocks are normalized to one. Other parameter values are given in Table 1.
Figure 3: Impulse Responses: (A) Ratio of net exports to GDP, (B) Ratio of consumption to GDP, (C) Ratio of investment to GDP

Notes: For all specifications, we set $\sigma = 2$, $\beta = 0.98$, and the standard deviations of technology shocks are normalized to one. All other parameters as shown in Table 1. We then produce the impulse responses of (A) ratio of net exports to GDP, (B) ratio of consumption to GDP, and (C) ratio of investment to GDP with different values of $\psi$: $\psi = \{0.001, 0.01, 2.8\}$. 
### Table 2: Moments - Emerging Market: Mexico

<table>
<thead>
<tr>
<th></th>
<th>AG</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>GPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi )</td>
<td>0.001</td>
<td>10(^{-12} )</td>
<td>0.00001</td>
<td>0.001</td>
<td>0.01</td>
<td>0.1</td>
<td>1</td>
<td>2.8</td>
</tr>
</tbody>
</table>

**Notes:** AG refers to specification 1 in Table 5 of AG and GPU refers to Garcia-Cicco, Pancrazi, and Uribe (2010). In GPU, \( \psi \) is estimated to be 2.8. Thus, we keep all other parameters as same as in AG and choose \( \psi \) to be 2.8. Similarly, for our specifications (1) to (6) are obtained by using different parameter values of \( \psi \).

moments. A value of \( \psi \) of around 0.1 already generates important difficulties at this task. Larger values overturn the results in AG.

### 4.2 Implications of Separable Preferences

Second, we investigate whether alternative preferences to those originally used by AG do a better job at generating consumption volatility. This is indeed suggested by Proposition 1 because only when preferences are separable consumption has a random-walk behavior and thus jumps on impact to the long-run level of productivity implied by the trend shock. The parametrization in AG actually sets \( \sigma = 2 \), which implies Cobb-Douglass, non-separable preferences, which can possibly dampen the reaction of consumption on impact.

To investigate this point we simulate moments with \( \sigma = 1 \) (and letting all other features and parameter values in the original AG model, including the productivity parameters). Table 3 shows that using separable preferences generates an increase in the volatility of consumption to output by 9%. This finding confirm the intuition provided by Proposition 1. Also, this produces an increase in the volatility of net exports to output by 11%. There is slight fall in the correlation between net exports and output from -0.66 to -0.62.
Table 3: Moments - Separable Preferences Rather than Cobb-Douglas

<table>
<thead>
<tr>
<th></th>
<th>AG (σ = 2)</th>
<th>AG: Separable preferences (σ = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ(c)/σ(y)</td>
<td>1.26</td>
<td>1.35</td>
</tr>
<tr>
<td>σ(NX)/σ(y)</td>
<td>0.71</td>
<td>0.82</td>
</tr>
<tr>
<td>ρ(NX, y)</td>
<td>-0.66</td>
<td>-0.62</td>
</tr>
</tbody>
</table>

Notes: All parameters are those from AG, except for σ in the second column.

Should one then advocate the use of separable preference in this type of exercises? Actually, one important caveat of separable preferences is the negative comovement of labor supply and trend productivity due to the wealth effect. Figure 4 shows that, after a permanent shock, labor supply falls. The next subsection discusses this issue further.

![Permanent Shock: Labor Supply](image)

Figure 4: Impulse Responses: Labor Supply

Notes: All parameters are those from AG, except for σ, which is set to 1.

4.3 A Simple Specification

We showed previously that even though separable preferences allow to more easily obtain consumption and net exports volatility, labor supply falls. Recalling Jaimovich and Rebelo (2009), one immediate reaction is that the model requires more frictions to perform well on this dimension. However, an easier and interesting fix is to remove labor supply altogether, i.e., to consider an endowment economy. Notice however that this would also require removing investment from the model. From the point of view of Proposition 1 this is not a problem because its proof does not use any particular form of the economy’s supply side.

Table 4 shows the results. It compares the moments generated by the baseline emerging market calibration in AG to a calibration of the simple model we propose. The calibration is in Table 5. The calibrated simple moments does very well in replicating the moments in AG, with a larger ratio of the volatility of consumption
Table 4: Moments - Simple Model

<table>
<thead>
<tr>
<th></th>
<th>AG</th>
<th>Simple Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(c)/\sigma(y)$</td>
<td>1.26</td>
<td>1.55</td>
</tr>
<tr>
<td>$\sigma(NX)/\sigma(y)$</td>
<td>0.71</td>
<td>0.66</td>
</tr>
<tr>
<td>$\rho(NX, y)$</td>
<td>-0.66</td>
<td>-0.66</td>
</tr>
</tbody>
</table>

*Notes:* The first column (AG) is from Column of Table 4 in AG (equivalently, Specification 1, Mexico, Table 5). For the simple model, we use the following parameter values $\rho_g = 0.40$, $\rho_z = 0.95$, $\beta = 0.99$, $\psi = 0.0001$. Other parameter values are from Table 1. The simple model does not include labor supply nor capital (endowment economy).

to the volatility of output $\sigma(c)/\sigma(y)$, a slightly smaller ratio of the volatility of net exports to the volatility of output $\sigma(NX)/\sigma(y)$, and the same correlation between net exports and output $\rho(NX, y)$. So, one could recur to the simple model to match these moments in the data.

Table 5: Calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B/Y$</td>
<td>Steady state level of normalized debt</td>
</tr>
<tr>
<td>Persistence permanent shock</td>
<td>0.40</td>
</tr>
<tr>
<td>Persistence transitory shock</td>
<td>0.95</td>
</tr>
<tr>
<td>Standard dev. permanent shock</td>
<td>2.81</td>
</tr>
<tr>
<td>Standard dev. transitory shock</td>
<td>0.48</td>
</tr>
</tbody>
</table>

*Notes:* All other parameters are from AG. Specifically, we set $\beta = 0.98$ and $\psi = 0.001$.

To finalize, we note that the success of the endowment economy in generating the moments emphasized by AG should make one suspicious that the RBC structure has anything to do with the ability of trend shocks in explaining the data. Also, as we note below, the RBC model has difficulties in generating a powerful propagation of trend shocks.

### 4.4 Bayesian Results

Garcia-Cicco, Pancrazi, and Uribe (2010) perform a Bayesian estimation of an RBC model with trend shocks and financial frictions using a long sample for Argentina. They find strong evidence that financial frictions are better able to fit the data in several aspects. In particular, a variance decomposition exercise on the estimated model reveals that trend shocks account for a very small fraction of the movements.
in consumption, output and net exports, and that financial shocks are a much more important driver of all these variables. The question we ask here is whether imposing the restrictions suggested by Proposition 1 can to some extent overturn this result, helping trend shocks to explain the cycle.

We try to remain as close as possible to GPU, thus we augment the Euler equation \((AGb17)\) with a preference shock:

\[
0 = \left(\gamma(1 - \sigma) - 1\right)E\hat{c}' + (1 - \gamma)(1 - \sigma)E\hat{l}' + (\gamma(1 - \sigma) - 1)g
\]

\[
-\left(\gamma(1 - \sigma) - 1\right)\hat{c} - (1 - \gamma)(1 - \sigma)\hat{l} - \hat{q} + E\nu' - \nu
\]

where the variable \(\nu\) represents an exogenous and stochastic preference shock following the AR(1) process:

\[
\ln \nu_t = \rho \ln \nu_{t-1} + \epsilon'_\nu, \quad \epsilon'_\nu \sim \text{i.i.d. } N(0, \sigma^2_{\nu})
\]

Similarly, the interest rate equation \((AGb24)\) is augmented with a country premium shock:

\[
\hat{q} = -\psi\bar{Q}\hat{b}' - \bar{Q}\mu_t
\]

where the variable \(\mu\) represents an exogenous stochastic country premium shock following the AR(1) process:

\[
\ln \mu_t = \rho \ln \mu_{t-1} + \epsilon'_\mu, \quad \epsilon'_\mu \sim \text{i.i.d. } N(0, \sigma^2_{\mu})
\]

The resource constraint \((AGb20)\) is also modified with a domestic spending shock \(\hat{s}\):

\[
0 = \bar{Y} \bar{\Gamma} \hat{y} + \bar{Q} \bar{b}' + \bar{Q} \bar{B} (g + \hat{q}) - \bar{b} - \bar{X} \bar{\Gamma} \hat{x} - \bar{C} \bar{\Gamma} \hat{c} - \bar{S} \bar{\Gamma} \hat{s}
\]

where \(\hat{s}\) follows the AR(1) process:

\[
\ln(\hat{s}_t/s) = \rho_s(\hat{s}_{t-1}/s) + \epsilon'_s, \quad \epsilon'_s \sim \text{i.i.d. } N(0, \sigma^2_{s})
\]

Following Proposition 1, we impose that \(\psi\) is very small (as small as numerically possible) and preferences are separable (GPU use \(\psi = 2.8\)). As we have shown above, using a large \(\psi\) tends to diminish the propagation of trend shocks, so using a small \(\psi\) gives trend shocks a good shot at competing with other shocks.\(^{11}\)

We estimate the model through Bayesian methods and Argentine data on output growth, consumption growth, investment growth, and the trade balance-to-output ratio over the period 1900–2005. Following GPU closely, we estimate 12 structural

\(^{11}\)We do not use a discount factor \(\beta\) numerically close to 1, because this would be clearly at odds with the data (GPU use \(\beta = 0.9224\), consistent with an average real interest rate in Argentina of about 8.5%).
parameters of the augmented model, $\bar{y}, \sigma_g, \rho_g, \sigma_\alpha, \rho_\alpha, \sigma_\nu, \rho_\nu, \sigma_s, \rho_s, \sigma_\mu, \rho_\mu, \phi$, and four nonstructural parameters representing the standard deviations of i.i.d. measurement errors of the observables, $\sigma_y, \sigma_c, \sigma_i$, and $\sigma_{tby}$. We impose uniform prior distributions on all estimated parameters.

Table 6 reports key statistics of prior and posterior distributions of estimated parameters. There are a number of interesting features to observe: First, GPU suggests a weak identification of the trend shock to productivity by highlighting quite diverse posterior distributions of the estimated parameters $\sigma_g$ and $\rho_g$. However, both parameters are rather well identified with the restrictions suggested by Proposition 1. Specifically, the posterior distributions of the parameters $\sigma_g$ and $\rho_g$ with 90 percent probability intervals are $(0.076, 0.098)$ and $(-0.120, -0.012)$. Second, the mean of estimated $\sigma_g$ takes the value 0.087, which is very close to the corresponding point estimate in the baseline RBC model under the proposed specification ($\sigma_g = 0.086$). This is a stark contrast to GPU where they find that $\sigma_g$ is estimated to be four times smaller with the introduction of financial frictions and preference shocks. Third, different from GPU, we find that the estimated volatility of the country-premium shock is not so large. The mean value of $\sigma_\mu$ is 0.007, which is substantially smaller than the corresponding point estimate in GPU (0.056). This implies that a one-standard deviation innovation in $\mu_t$ raises the interest rate at which the country borrows from world financial markets by less than one-percentage point.

Table 7 reports the percentage of forecast error of output, consumption, investment, and trade balance (variance decomposition) explained by nonstationary technology shocks. We compare across model specifications. In the GPU specification, we set $\sigma$ to 2 and relax the imposition that $\psi$ is very small. It shows that trend shocks account for a very small fraction of the movements in consumption, output, and net exports, similar to the findings of GPU (Table 5, p. 2525.) In Lines 2 to 5, we estimate the model specified in GPU but, following Proposition 1, we set $\psi$ to a very small value ($10^{-12}$) and $\sigma$ to 1 (separable preferences). In Line 2, we allow, as GPU, for a domestic preference shock, a spending shocks, and a country premium shock. In Line 3, only the domestic preference shock and the spending shock are present; In Line 4, only the country premium shock and a spending shock are present; In Line 5, only the spending shock is present. The results suggest that domestic preference and country premium shocks explain a very large fraction of business cycle volatility. In fact, in Line 2, preference and country premium shocks each account for 62.2% and 23.9% of output volatility respectively, 69.6% and 29.8% of consumption volatility respectively, 35.0% and 62.9% of investment volatility, and for 98.8% and 1.1% of

\textsuperscript{12}In GPU, the posterior distributions of of the parameters $\sigma_g$ and $\rho_g$ with 90 percent probability intervals were $(0.00057, 0.027)$ and $(-0.66, 0.83)$ (GPU Table 3, p. 2518).
## Table 6: Prior and Posterior Distributions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Min</th>
<th>Max</th>
<th>Prior distribution</th>
<th>Posterior distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{g}$</td>
<td>1</td>
<td>1.03</td>
<td>1.007</td>
<td>1.000</td>
</tr>
<tr>
<td>$\sigma_g$</td>
<td>0</td>
<td>0.20</td>
<td>0.087</td>
<td>0.076</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>-0.99</td>
<td>0.99</td>
<td>-0.067</td>
<td>-0.120</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>0</td>
<td>0.20</td>
<td>0.010</td>
<td>0.001</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>-0.99</td>
<td>0.99</td>
<td>0.667</td>
<td>0.602</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0</td>
<td>8</td>
<td>2.2</td>
<td>1.5</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0</td>
<td>1.00</td>
<td>0.015</td>
<td>0.000</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>-0.99</td>
<td>0.99</td>
<td>0.977</td>
<td>0.971</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0</td>
<td>0.20</td>
<td>0.020</td>
<td>0.000</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>-0.99</td>
<td>0.99</td>
<td>0.183</td>
<td>-0.460</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>0</td>
<td>0.20</td>
<td>0.007</td>
<td>0.003</td>
</tr>
<tr>
<td>$\rho_\mu$</td>
<td>-0.99</td>
<td>0.99</td>
<td>0.074</td>
<td>-0.416</td>
</tr>
</tbody>
</table>

### Measurement errors

| $\sigma^2_y$ | 0.01 | 0.114 | 0.0125 | 0.0100 | 0.0161 |
| $\sigma^2_c$ | 0.01 | 0.138 | 0.0293 | 0.0249 | 0.0336 |
| $\sigma^2_i$ | 0.01 | 0.226 | 0.1495 | 0.1305 | 0.1670 |
| $\sigma^2_{tby}$ | 0.01 | 0.114 | 0.0111 | 0.0100 | 0.0127 |

**Notes:** Estimation is based on Argentine data from 1990 to 2005. All prior distributions are uniform. Posterior statistics are based on a 200000 MCMC chain from which the first 100000 draws were discarded. The estimated measurement errors are standard deviations.

*net exports volatility respectively.*

Trend shocks, instead, become relevant only to explain consumption and investment volatilities when both of these shocks are absent from the model as shown in Line 5. Notice that in this case however, trend shocks still have a hard time propagating to net exports and output.

To sum up, Bayesian estimation establishes empirically that, even when we try to stick as close as possible to the premise of Proposition 1, the RBC model does a poor job propagating trend shocks to net exports and output. Financial and preference shocks explain the lion’s share of net exports and output volatility.

## 5 Conclusions

Two main conclusions can be drawn from these exercises. The first concerns the importance of efforts towards a precise and well-identified estimation of the sensitivity of the interest rate $\psi$. Recently Miyamoto and Nguyen (2015) estimate using data for 17 developing and developed countries and find that $\psi$ significantly different from 0.
Table 7: Percentage of Forecast Error Variance of Selected Endogenous Variables Predicted for Non-stationary Technology Shocks, Comparing Across Specifications

| 1. GPU specification | 2.55 | 1.61 | 0.34 | 0.22 |
| 2. Limit specification | 0.15 | 0.34 | 0.22 | 0.00 |
| 3. Limit (without country premium shock) | 0.18 | 0.47 | 0.59 | 0.00 |
| 4. Limit (without pref. shock) | 0.38 | 1.01 | 0.29 | 1.11 |
| 5. Limit (without pref. and country premium shocks) | 1.01 | 64.72 | 14.25 | 2.89 |

Notes: Table presents the forecast error variance decomposition of output growth, consumption growth, investment growth, and net exports, reporting the contribution of nonstationary technology (or trend) shocks. In all cases, we set $\beta$ to 0.9224. Means are based on 200,000 draws from the posterior distribution.

and differs across countries. So far, these estimates have been obtained via structural estimation. Therefore, identification has remained dependent on the exact specification and details of the model. Finding alternative and more direct approaches to identifying the sensitivity of the interest rate is a fruitful research avenue.

The second conclusion is the serious analytical and quantitative difficulties in using the RBC model to analyze the propagation of trend shocks. Analytically, this requires separable preferences, which immediately introduces counterfactual labor input dynamics. Quantitatively, an estimation of the RBC model augmented with preference, country premium, and spending shocks favors financial frictions and other disturbances over trend shocks as an explanation of the data, even when the analytical conditions are imposed. This leads us to conclude that the RBC framework is not well suited to generate realistic emerging markets business cycles.

Notice however that these are not statements about the ability of trend shocks to fit the data in other models. First, trend shocks have an important conceptual advantage, which is of resting on a well-established economic mechanism (the permanent income hypothesis) in order to (at least qualitatively) generate highly volatile consumption and net exports, and countercyclical net exports. Moreover, trend shocks have been successfully used in “cousin” literatures to match interesting facts implied by consumption dynamics. Thus, we remain under the impression that efforts in the direction of specifying open economy models with extra frictions that allow trend shocks to propagate in a realistic way constitute a fruitful research avenue. We look forward to developments in this direction.

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14See, for instance, Blanchard et al. (2013).
A  Main Proof

The following equations help determine the steady state values and will be important for the proof of Proposition 1:

\[
\begin{align*}
\bar{Q} &= \frac{1}{1 + r^*} = \beta \\
(1 - \alpha) \frac{\bar{Y}}{\bar{K}} &= \frac{1}{Q} - 1 + \delta \\
\bar{C} \frac{\bar{Y}}{\bar{Y}} &= 1 - \delta \frac{\bar{K}}{\bar{Y}} + (\bar{Q} - 1) \frac{\bar{B}}{\bar{Y}} \\
\bar{N} &= \left(1 + \frac{\bar{C}}{\bar{Y}} \left(\frac{1 - \gamma}{\alpha \gamma}\right)\right)^{-1} \\
\frac{\bar{K}}{\Gamma} &= \left(\frac{\bar{K}}{\bar{Y}}\right)^{\frac{1}{\alpha}} \bar{N} \\
\frac{\bar{Y}}{\Gamma} &= \left(\frac{\bar{Y}}{\bar{K}}\right) \left(\frac{\bar{K}}{\Gamma}\right) \\
\frac{\bar{C}}{\Gamma} &= \left(\frac{\bar{C}}{\bar{Y}}\right) \left(\frac{\bar{Y}}{\Gamma}\right) \\
\frac{\bar{X}}{\Gamma} &= \delta \frac{\bar{K}}{\Gamma} \\
\bar{N}X/\bar{Y} &= 1 - \frac{\bar{C}}{\bar{Y}} - \frac{\bar{X}}{\bar{Y}}
\end{align*}
\]

Proof of Proposition 1. Substituting \(\hat{y}_t\) from (2) and substituting \(\hat{x}_t\) from (AGb21) in the linearization of the budget constraint (AGb20) implies

\[
0 = \frac{\bar{Y}}{\Gamma} \left(z_t + (1 - \alpha) \hat{k}_t + \alpha g_t\right) + \bar{Q} \frac{\bar{B}}{\Gamma} \left(g_t - \psi \bar{Q} \hat{b}_{t+1}\right)
+ Q\bar{b}_{t+1} - \hat{b}_t - \frac{\bar{K}}{\Gamma} \left(\hat{k}_{t+1} - (1 - \delta) \hat{k}_t + g_t\right) - \bar{C} \frac{\hat{e}_t}{\Gamma}
\]
Combining with the definition of $b_{t+1}$ and $c_t$, we obtain

$$Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right) b_{t+1} = b_t - \left( \alpha \frac{Y}{\Gamma} - \frac{\bar{K}}{\Gamma} + Q \frac{\bar{B}}{\Gamma} \right) y_t - \left( \frac{\bar{Y}}{\Gamma} (1 - \alpha) + \frac{\bar{K}}{\Gamma} (1 - \delta) \right) \hat{k}_t - \frac{\bar{Y}}{\Gamma} z_t - \frac{\bar{C}}{\Gamma} \gamma_{t-1} + Q \frac{\bar{B}}{\Gamma} \left(1 - \psi Q \frac{\bar{B}}{\Gamma} \right) \gamma_t - \frac{\bar{B}}{\Gamma} \gamma_{t-1} + \left[ \frac{\bar{C}/\Gamma}{\bar{K}/\Gamma} \right] c_t \hat{k}_{t+1}$$

$$= \left[ 1 - \left( \frac{\bar{Y}}{\Gamma} (1 - \alpha) + \frac{\bar{K}}{\Gamma} (1 - \delta) \right) \right] X_t^0 + \left[ - \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \psi \left( \frac{\bar{Q} \bar{B}}{\Gamma} \right)^2 \right) \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \bar{Q} \frac{\bar{B}}{\Gamma} - \frac{\bar{C}}{\Gamma} \right) - \frac{\bar{Y}}{\Gamma} \right] X_t^1 + \frac{\bar{C}}{\Gamma} D_c^0 X_t^0 + \frac{\bar{K}}{\Gamma} D_k^0 X_t^0 + \frac{\bar{C}}{\Gamma} D_c^1 X_t^1 + \frac{\bar{K}}{\Gamma} D_k^1 X_t^1$$

We regroup the coefficients on $X_t^0$ and $X_t^1$ to obtain:

$$D_b^0 = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( D_b + \frac{\bar{C}}{\Gamma} D_c^0 + \frac{\bar{K}}{\Gamma} D_k^0 \right)$$

where $D_b = \left[ 1 - \left( \frac{\bar{Y}}{\Gamma} (1 - \alpha) + \frac{\bar{K}}{\Gamma} (1 - \delta) \right) \right]$ and

$$D_b^1 = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left[ - \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \psi \left( \frac{\bar{Q} \bar{B}}{\Gamma} \right)^2 \right) \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + (\bar{Q} - 1) \frac{\bar{B}}{\Gamma} - \frac{\bar{C}}{\Gamma} \right) - \frac{\bar{Y}}{\Gamma} \right] + \frac{\bar{C}}{\Gamma} D_c^1 + \frac{\bar{K}}{\Gamma} D_k^1$$

More explicitly:

$$D_{b,b} = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( 1 + \frac{\bar{K}}{\Gamma} D_{k,b} + \frac{\bar{C}}{\Gamma} D_{c,b} \right)$$

$$D_{b,k} = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( \frac{\bar{K}}{\Gamma} D_{k,k} + \frac{\bar{C}}{\Gamma} D_{c,k} - \left( \frac{\bar{Y}}{\Gamma} (1 - \alpha) + \frac{\bar{K}}{\Gamma} (1 - \delta) \right) \right)$$

and

$$D_{b,z} = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( \frac{\bar{K}}{\Gamma} D_{k,z} + \frac{\bar{C}}{\Gamma} D_{c,z} - \frac{\bar{Y}}{\Gamma} \right)$$

$$D_{b,\zeta 1} = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( \frac{\bar{K}}{\Gamma} D_{k,\zeta 1} + \frac{\bar{C}}{\Gamma} D_{c,\zeta 1} - \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \psi \left( \frac{\bar{Q} \bar{B}}{\Gamma} \right)^2 \right) \right)$$

$$D_{b,\zeta 2} = \frac{1}{Q \left(1 - \psi \frac{\bar{B}}{\Gamma} \right)} \left( \frac{\bar{K}}{\Gamma} D_{k,\zeta 2} + \frac{\bar{C}}{\Gamma} D_{c,\zeta 2} + \left( \alpha \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + Q \frac{\bar{B}}{\Gamma} - \frac{\bar{C}}{\Gamma} - \frac{\bar{B}}{\Gamma} \right) \right)$$

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From the log-linearizing equation of the Euler equation with respect to $k'$ (AGb12), using the assumption that $\gamma = 1$ and $\hat{\ell} \equiv 0$, we have

\[
0 = -\sigma E_t [c_{t+1} - c_t] + \beta \phi E_t [g_{t+1}]
+ \beta (1 - \alpha) \frac{\bar{Y}}{K} E_t [z_{t+1} + (1 - \alpha) \hat{k}_{t+1} + \alpha g_{t+1}] + \beta \phi E_t [\hat{k}_{t+2}]
- \left( \beta \left( (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) + \phi \right) \hat{k}_{t+1} - (\sigma + \phi) g_t + \phi \hat{k}_t
\]

(8)

We use the fact that

\[
E_t [g_{t+1}] = \rho g_t
\]
\[
E_t [z_{t+1}] = \rho z_t
\]

and $g_t = \zeta - \zeta_{t-1}$, together with

\[
E_t [\hat{k}_{t+2}] = E_t [D_k X_{t+1}]
= E_t [D_{c,b} D_b X_t + D_{k,k} D_k X_t + D_{k,z} z_{t+1} + D_{k,z1} \zeta_{t+1}] + D_{k,\zeta2} \zeta_t
= (D_{c,b} D_b + D_{c,k} D_k) X_t + D_{k,z} z_t + ((1 + \rho_g) D_{c,z1} + D_{c,\zeta2}) \zeta_t - \rho_g D_{c,\zeta1} \zeta_{t-1}
\]
\[
E_t [\hat{k}_{t+1}] = D_k X_t
\]
\[
E_t [c_{t+1}] = E_t [D_c X_{t+1}]
= (D_{c,b} D_b + D_{c,k} D_k) X_t + D_{c,z} z_t + ((1 + \rho_g) D_{c,z1} + D_{c,\zeta2}) \zeta_t - \rho_g D_{c,\zeta1} \zeta_{t-1}
\]
\[
E_t [c_t] = D_c X_t
\]

to simplify (8) to

\[
0 = -\sigma ((D_{c,b} D_b + D_{c,k} D_k) X_t + D_{c,z} z_t + ((1 + \rho_g) D_{c,z1} + D_{c,\zeta2}) \zeta_t - \rho_g D_{c,\zeta1} \zeta_{t-1} - D_c X_t)
+ \beta \phi \rho_g (\zeta - \zeta_{t-1}) + \beta (1 - \alpha) \frac{\bar{Y}}{K} (\rho z_t + \alpha \rho_g (\zeta_t - \zeta_{t-1})) + \beta (1 - \alpha) \frac{\bar{Y}}{K} (1 - \alpha) D_k X_t
+ \beta \phi ((D_{k,b} D_b + D_{k,k} D_k) X_t + D_{k,z} z_t + ((1 + \rho_g) D_{k,z1} + D_{k,\zeta2}) \zeta_t - \rho_g D_{k,\zeta1} \zeta_{t-1})
- \left( \beta \left( (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) + \phi \right) D_k X_t - (\sigma + \phi) (\zeta_t - \zeta_{t-1}) + \phi \hat{k}_t
\]
Now, extracting the components related to \( X^0_t \) from this equation, we have

\[
0 = -\sigma ((D_{c,b}D^0_b + D_{c,k}D^0_k) X^0_t - D^0_c X^0_t) \\
+ \beta (1 - \alpha) \frac{\bar{Y}}{K} (1 - \alpha) D^0_b X^0_t + \beta \phi ((D_{k,b}D^0_b + D_{k,k}D^0_k) X^0_t) \\
- \left( \beta \left( (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) + \phi \right) D^0_k X^0_t + \phi k_t
\]

for all \( X^0_t \). This implies

\[
0 = -\sigma ((D_{c,b}D^0_b + D_{c,k}D^0_k) - D^0_c) \\
+ \beta (1 - \alpha) \frac{\bar{Y}}{K} (1 - \alpha) D^0_b + \beta \phi (D_{k,b}D^0_b + D_{k,k}D^0_k) \\
- \left( \beta \left( (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) + \phi \right) D^0_k + \phi D_k
\]

(9)

where \( D_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \). This equation helps determines \( D^0_k \), i.e., \( D_{k,b} \) and \( D_{k,k} \) as functions of \( D_{c,b} \) and \( D_{c,k} \). In particular when \( D_{c,b} \) and \( D_{c,k} \) are close to zero, we have the Taylor expansion:

\[
D_{k,b} = \alpha_1 D_{c,b} + \beta_1 D_{c,k} + o(D_{c,b}) + o(D_{c,k}) \\
D_{k,k} = D^*_{k,k} + \alpha_2 D_{c,b} + \beta_2 D_{c,k} + o(D_{c,b}) + o(D_{c,k})
\]

(10)

where \( D^*_{k,k} \) is the solution of

\[
0 = - \left( \beta \left( (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) + \phi \right) D^*_{k,k} + \beta \phi (D^*_{k,k})^2 + \phi
\]

i.e. equation (9) for \( D^0_{k,k} \) when \( D^0_c = 0 \).

Armed with the solution (10), we now use the first order condition for \( b' \) (AGb17), again with \( \gamma = 1 \) and \( \tilde{l} \equiv 0 \):

\[
0 = \sigma \mathbb{E}_t [c_{t+1} - c_t] - \psi \bar{Q} b_{t+1} + \psi \bar{B} \frac{\bar{F}}{\bar{I}} \zeta_t
\]

(11)

and extract the coefficients on \( X^0_t \) to obtain

\[
0 = \sigma ((D_{c,b}D^0_b + D_{c,k}D^0_k) - D^0_c) - \psi \bar{Q} D^0_b
\]

Substituting \( D^0_b \) from (7) into this equation, we arrive at
\[ 0 = \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \left( \bar{B}_{c,b} + \bar{C} \bar{D}_{c}^0 + \bar{D}_{k,b}^0 \right) + D_{c,k}D_{k,b}^0 - D_c^0 \]

We separate the equations for \( D_{c,b} \) and \( D_{c,k} \) to obtain

\[ 0 = \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \left( \bar{B}_{b,b} + \bar{C} \bar{D}_{c,b} \right) - D_{c,b} \]
\[ + \left\{ \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \bar{K} + D_{c,k} \right\} D_{k,b} \] (12)

and

\[ 0 = \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \left( \bar{B}_{b,k} + \bar{C} \bar{D}_{c,k} \right) - D_{c,k} \]
\[ + \left\{ \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \bar{K} + D_{c,k} \right\} D_{k,k} \] (13)

Now, we use the results above to show that as \( \bar{Q} \to 1 \) and \( \psi \to 0 \), \( D_{c,b} \to 0 \).

Indeed, we first solve for \( D_{c,k} \) from the second equation (13):

\[ D_{c,k} = \frac{\left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \left( \bar{B}_{b,k} + \bar{K} \bar{D}_{k,k} \right)}{1 - \left( D_{c,b} - \frac{\psi}{\sigma}Q \right) \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{\Gamma} \right)} \frac{\bar{C}}{\Gamma} - D_{k,k}} \]

As \( \psi \to 0 \), this equation simplifies to

\[ D_{c,k} = \frac{D_{c,b} \frac{1}{\bar{Q}} \left( \bar{B}_{b,k} + \bar{K} \bar{D}_{k,k} \right)}{1 - D_{c,b} \frac{1}{\bar{Q}} \frac{\bar{C}}{\Gamma} - D_{k,k}} \] (14)

In addition, equation (12) simplifies to

\[ 0 = D_{c,b} \frac{1}{\bar{Q}} \left( \bar{B}_{b,b} + \bar{C} \bar{D}_{c,b} \right) - D_{c,b} + \left\{ D_{c,b} \frac{1}{\bar{Q}} \bar{K} + D_{c,k} \right\} D_{k,b} \] (15)

Plugging (14) into (15) and grouping by \( D_{c,b} \) (also by definition \( \bar{B}_{b,b} = 1 \)), we
have

\[ 0 = \frac{1}{Q} \left( 1 + \frac{\bar{C}}{\Gamma} D_{c,b} \right) - 1 + \left\{ \frac{1}{Q} \frac{\bar{K}}{\Gamma} + \frac{1}{Q} \left( \frac{D_{b,k}}{\Gamma} + \frac{\bar{K}}{\Gamma} D_{k,k} \right) \right\} D_{k,b} \]

Equivalently,

\[ \bar{Q} - 1 = \frac{\bar{C}}{\Gamma} D_{c,b} + \left\{ \frac{\bar{K}}{\Gamma} + \frac{D_{b,k}}{\Gamma} \frac{\bar{C}}{\Gamma} - D_{k,k} \right\} D_{k,b} \]

As \( \bar{Q} \to 1 \) and \( \psi \to 0 \), Lemma 1 below shows that (10) holds with \( \alpha_1 = \beta_1 = 0 \), and \( 0 < D^*_k,k < 1 \). Therefore,

\[ \bar{Q} - 1 = \frac{\bar{C}}{\Gamma} D_{c,b} + o(D_{c,b}) \]

Therefore, as \( \bar{Q} \to 1 \), \( D_{c,b} \to 0 \).

Then, (10) and (14) imply that

\[
\begin{align*}
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,k} &= 0 \\
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{k,b} &= 0 \\
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{k,k} &= \lim_{\bar{Q} \to 1} D^*_k,k \\
&= \frac{(\alpha \delta / \phi + 2) - \sqrt{(\alpha \delta / \phi + 2)^2 - 4}}{2}
\end{align*}
\]

where the last limit is given in Lemma 1.

We now move on to compute \( D^1_c \).

Rearranging equation (11) and using the conjecture for \( c_{t+1} \), we obtain:

\[ 0 = \left( D_{c,b} - \frac{\psi \bar{Q}}{\sigma} \right) b_{t+1} + D_{c,k} \hat{k}_{t+1} + D^1_c A \mathbf{x}^1_t - c_t + \frac{\psi \bar{Q} \bar{B}}{\sigma} \zeta_t \]

and substituting \( b_{t+1} \) from (6), we have

\[
(1 - \bar{x}) c_t = \frac{\bar{x}}{C/\Gamma} \left[ b_t - \left( \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \psi \left( \frac{\bar{Q} \bar{B}}{\Gamma} \right)^2 \right) \zeta_t + \left( \frac{\bar{Y}}{\Gamma} - \frac{\bar{K}}{\Gamma} + \frac{\bar{Q} \bar{B}}{\Gamma} - \frac{\bar{C}}{\Gamma} - \frac{\bar{B}}{\Gamma} \right) \zeta_{t-1} - \frac{\bar{Y}}{\Gamma} \bar{z}_t \right] \\
- \frac{\bar{x}}{C/\Gamma} \left[ \left( \frac{\bar{Y}}{\Gamma} (1 - \alpha) + \frac{\bar{K}}{\Gamma} (1 - \delta) \right) \hat{k}_t - \frac{\bar{K}}{\Gamma} \hat{k}_{t+1} \right] \\
+ D^1_c A \mathbf{x}^1_t + D_{c,k} \hat{k}_{t+1} + \frac{\psi \bar{Q} \bar{B}}{\sigma} \zeta_t \tag{16}
\]
where

\[ A = \begin{bmatrix} 1 + \rho_g & -\rho_g & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho_z \end{bmatrix} \]

and

\[ \bar{x} = \frac{(D_{c,b} - \frac{\psi \bar{Q}}{\sigma})\bar{C}/\bar{\Gamma}}{\bar{Q} \left(1 - \psi \bar{Q} \frac{\bar{B}}{\bar{F}}\right)} \]

Collecting the terms for \( \zeta_t \) from (16), we have

\[
(1 - \bar{x}) D_{c,\zeta 1} = (1 + \rho_g)D_{c,\zeta 1} + D_{c,\zeta 2} + \frac{\psi \bar{Q} \bar{B}/\bar{\Gamma}}{\sigma} \\
- \frac{\bar{x}}{C/\bar{\Gamma}} \left(\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi \left(\frac{\bar{Q} \bar{B}}{\bar{\Gamma}}\right)^2\right) \\
+ \left(\frac{\bar{x}}{C/\bar{\Gamma}} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right)D_{k,\zeta 1}
\]

which leads to

\[
(\rho_g + \bar{x})D_{c,\zeta 1} + D_{c,\zeta 2} = \frac{\bar{x}}{C/\bar{\Gamma}} \left(\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \psi \left(\frac{\bar{Q} \bar{B}}{\bar{\Gamma}}\right)^2\right) \\
- \left(\frac{\bar{x}}{C/\bar{\Gamma}} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right)D_{k,\zeta 1} - \frac{\psi \bar{Q} \bar{B}/\bar{\Gamma}}{\sigma} \tag{17}
\]

Similarly, collecting the terms for \( \zeta_{t-1} \) from (16), we have

\[
(1 - \bar{x}) D_{c,\zeta 2} = -\rho_g D_{c,\zeta 1} + \frac{\bar{x}}{C/\bar{\Gamma}} \left(\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q} \frac{\bar{B}}{\bar{\Gamma}} - \bar{C} \frac{\bar{B}}{\bar{\Gamma}}\right) \\
+ \left(\frac{\bar{x}}{C/\bar{\Gamma}} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right)D_{k,\zeta 2}
\]

which leads to

\[
\frac{\rho_g}{1 - \bar{x}} D_{c,\zeta 1} + D_{c,\zeta 2} = \frac{\bar{x}}{1 - \bar{x}} \left(\frac{1}{C/\bar{\Gamma}}\right) \left(\frac{\bar{Y}}{\bar{\Gamma}} - \frac{\bar{K}}{\bar{\Gamma}} + \bar{Q} \frac{\bar{B}}{\bar{\Gamma}} - \bar{C} \frac{\bar{B}}{\bar{\Gamma}}\right) \\
+ \frac{1}{1 - \bar{x}} \left(\frac{\bar{x}}{C/\bar{\Gamma}} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k}\right)D_{k,\zeta 2} \tag{18}
\]

Substituting \( D_{c,\zeta 2} \) from (18) into (17) and using the following steady state relations:

\[
\frac{\bar{C}}{\bar{\Gamma}} + \frac{\bar{X}}{\bar{\Gamma}} + \frac{\bar{B}}{\bar{\Gamma}} = \frac{\bar{Y}}{\bar{\Gamma}} + \bar{Q} \frac{\bar{B}}{\bar{\Gamma}}
\]

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we obtain

\[
D_{c,\zeta_1} = \left( \frac{1 - \bar{x}}{1 - \bar{x} - \rho_g} \right) \left( \frac{1}{C/\bar{\Gamma}} \right) \left( \frac{\bar{\alpha} \bar{Y}}{\bar{\Gamma}} - \frac{\bar{\bar{K}}}{\bar{\Gamma}} + \psi \left( \frac{\bar{Q} \bar{B}}{\bar{\Gamma}} \right)^2 \right) \\
- \left( \frac{1}{1 - \bar{x} - \rho_g} \right) \left( \frac{1}{C/\bar{\Gamma}} \right) \left( \frac{\bar{\alpha} \bar{Y}}{\bar{\Gamma}} - \frac{\bar{\bar{K}}}{\bar{\Gamma}} + \bar{\bar{X}} - \bar{\bar{Y}} \right) \\
- \left( \frac{1 - \bar{x}}{1 - \bar{x} - \rho_g} \right) \left( \frac{1}{C/\bar{\Gamma}} \right) \left( \frac{1}{C/\bar{\Gamma}} \right) \left( D_{k,\zeta_1} + D_{k,\zeta_2} \right) \\
- \left( \frac{1}{1 - \bar{x} - \rho_g} \right) \left( \frac{1 - \bar{x}}{\bar{x}} \right) \left( \frac{1}{C/\bar{\Gamma}} \right) D_{c,k} \left( D_{k,\zeta_1} + D_{k,\zeta_2} \right) \\
- \left( \frac{1}{1 - \bar{x} - \rho_g} \right) \left( \frac{\bar{x}}{C/\bar{\Gamma}} \right) \left( \frac{\bar{\bar{K}}}{\bar{\Gamma}} \right) + D_{c,k} \right) D_{k,\zeta_2} 
\]  

(19)

Now as \( \psi \) goes to zero,

\[
\lim_{\psi \to 0} \bar{x} = \left( \frac{D_{c,b}}{\bar{Q}} \right) \left( \bar{C}/\bar{\Gamma} \right)
\]

Then, as \( \bar{Q} \) goes to one, as shown above, we have

\[
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} \bar{x} = \lim_{\bar{Q} \to 1} \lim_{\psi \to 0} \left( \frac{D_{c,b}}{\bar{Q}} \right) \left( \bar{C}/\bar{\Gamma} \right) = 0
\]

In addition, Lemma 2 shows that as \( \bar{Q} \to 1 \) and \( \psi \to 0 \), \( D_{k,\zeta_1} + D_{k,\zeta_2} = 0 \). Therefore (19) implies that

\[
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,\zeta_1} = \left( \frac{1 - \bar{X}/\bar{Y}}{C/\bar{Y}} \right) \left( \frac{1}{1 - \rho_g} \right)
\]

Similarly, from (18),

\[
\lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,\zeta_2} = -\rho_g \lim_{\bar{Q} \to 1} \lim_{\psi \to 0} D_{c,\zeta_1} \\
= \left( \frac{1 - \bar{X}/\bar{Y}}{C/\bar{Y}} \right) \left( \frac{-\rho_g}{1 - \rho_g} \right)
\]

Finally, collecting the terms for \( z_t \) from (16), we have

\[
(1 - \bar{x}) D_{c,z} = \rho_z D_{c,z} - \frac{\bar{x}}{C/\bar{\Gamma}} \left( \frac{\bar{Y}}{\bar{\Gamma}} \right) + \left( \frac{\bar{x}}{C/\bar{\Gamma}} \frac{\bar{K}}{\bar{\Gamma}} + D_{c,k} \right) D_{k,z}
\]
and rearranging the equations, we have

\[
D_{c,z} = \frac{1}{(1 - \bar{x} - \rho_z)} \left( -\frac{\bar{x}}{C/\Gamma} \left( \frac{\bar{Y}}{\Gamma} \right) + \left( \frac{\bar{x}}{C/\Gamma} \left( \frac{\bar{K}}{\Gamma} \right) + D_{c,k} \right) D_{k,z} \right)
\]

As \(\psi\) goes to zero and \(\bar{Q}\) goes to one, we have already shown that \(\bar{x}\) goes to zero and that \(D_{c,k}\) goes to zero such that in the limit \(D_{c,z}\) becomes

\[
\lim_{\bar{Q} \to 1, \psi \to 0} D_{c,z} = \frac{1}{1 - \rho_g} \times 0 = 0
\]

This completes the proof.

\textbf{Lemma 1} Consider the Taylor expansion in (10). As \(\bar{Q} \to 1\) and \(\psi \to 0\), \(\alpha_1, \beta_1 \to 0\), and \(0 < D^*_{k,k} < 1\).

\textbf{Proof.} First of all, from the equation that determines \(D^*_{k,k}\) in (10), as \(\bar{Q} \to 1\) and \(\psi \to 0\), this equation becomes

\[
0 = (D^*_{k,k})^2 - \left( \frac{\alpha \delta}{\phi} + 2 \right) D^*_{k,k} + 1
\]

since \((1 - \alpha)\frac{\bar{Y}}{K} = \delta\). This equation gives

\[
D^*_{k,k} = \frac{(\alpha \delta / \phi + 2) - \sqrt{(\alpha \delta / \phi + 2)^2 - 4}}{2} = \frac{2}{(\alpha \delta / \phi + 2) + \sqrt{(\alpha \delta / \phi + 2)^2 - 4}} \in (0, 1)
\]

Now, we can use the solution for \(D_{k,b}\) and \(D_{k,k}\) from (10) to obtain the constant \(\alpha_1\) and \(\beta_1\). We collect the terms for \(b_t\) from (9):

\[
0 = -\sigma \left( (D_{c,b}D_{b,b}) - D_{c,b} \right) + \beta \delta (1 - \alpha) D_{k,b} + \beta \phi (D_{k,b}D_{b,b} + D_{k,k}D_{k,b}) - (\beta (\delta + \phi) + \phi) D_{k,b}
\]

Substituting \(D_{b,b}\) from (7) and let \(\hat{D}_{k,k} = D_{k,k} - D^*_{k,k}\), we rewrite the last equation
as

\[ 0 = -\sigma \left( D_{c,b} \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} \left( 1 + \frac{K}{\Gamma} D_{k,b} + \frac{C}{\Gamma} D_{c,b} \right) \right) - D_{c,b} \right) + \beta \delta (1 - \alpha) D_{k,b} \]

\[ + \beta \phi \left( D_{k,b} \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} \left( 1 + \frac{K}{\Gamma} D_{k,b} + \frac{C}{\Gamma} D_{c,b} \right) \right) + \left( \tilde{D}_{k,k} + D_{k,b}^* \right) D_{k,b} \right) \]

\[ - (\beta (\delta + \phi) + \phi) D_{k,b} \]

Ignoring the second order terms such as \( D_{c,b} D_{k,b}, D_{c,b}^2, D_{k,b}^2 \), and \( \tilde{D}_{k,k} D_{k,b} \) in the first order approximation, the last equation becomes

\[ 0 = -\sigma \left( D_{c,b} \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} - D_{c,b} \right) + \beta \delta (1 - \alpha) D_{k,b} \]

\[ + \beta \phi \left( D_{k,b} \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} + D_{k,k}^* D_{k,b} \right) \right) - (\beta (\delta + \phi) + \phi) D_{k,b} \]

Therefore, we have

\[ \left( \beta (\delta + \phi) + \phi - \beta \delta (1 - \alpha) - \beta \phi \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} \right) - D_{k,k}^* \beta \phi \right) D_{k,b} = -\sigma \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} - 1 \right) D_{c,b} \]

or equivalently,

\[ \left( \beta \delta \alpha + \phi \left( 1 - \frac{\beta}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} \right) + \beta \phi \left( 1 - D_{k,k}^* \right) \right) D_{k,b} = -\sigma \left( \frac{1}{Q \left( 1 - \psi \bar{Q} \frac{B}{T} \right)} - 1 \right) D_{c,b} \]

Then, as \( \psi \) goes to zero, since \( \beta = \bar{Q} \), we have

\[ (\beta \delta \alpha + \beta \phi \left( 1 - D_{k,k}^* \right)) D_{k,b} = -\sigma \left( \frac{1}{\bar{Q}} - 1 \right) D_{c,b} \]

Also, \( \bar{Q} \rightarrow 1 \), the coefficient on \( D_{k,b} \) on the right-hand side is

\[ \delta \alpha + \phi \left( 1 - D_{k,k}^* \right) > 0 \]

and on the left-hand side

\[ \frac{1}{\bar{Q}} - 1 \rightarrow 0 \]
Therefore, $\alpha_1, \beta_1 \to 0$.

Lemma 2

As $\bar{Q} \to 1$ and $\psi \to 0$, $D_{k,\zeta_1} + D_{k,\zeta_2} = 0$.

Proof. Combining the Euler equation with respect to $k'$ (8) and the Euler equation with respect to $b'$ (11), we have

$$0 = \sigma(\zeta_t - \zeta_{t-1}) + \psi \bar{Q} b_{t+1} - \psi \bar{Q} \frac{\bar{B}}{\Gamma} \zeta_t$$
$$+ \beta \phi \rho_g (\zeta_t - \zeta_{t-1}) + \beta (1 - \alpha) \frac{\bar{Y}}{K} \left( \rho_z z_t + (1 - \alpha) \dot{k}_{t+1} + \alpha \rho (\zeta_t - \zeta_{t-1}) \right)$$
$$+ \beta \phi \mathbb{E}[\dot{k}_{t+2} - \dot{k}_{t+1}]$$
$$- \left( \beta (1 - \alpha) \frac{\bar{Y}}{K} + \phi \right) \dot{k}_{t+1} - (\sigma + \phi)(\zeta_t - \zeta_{t-1}) + \phi \dot{k}_t$$

As $\psi$ goes to zero, this equation simplifies to

$$0 = \left( \beta \phi \rho_g + \beta (1 - \alpha) \frac{\bar{Y}}{K} \alpha \rho_g - \phi \right) (\zeta_t - \zeta_{t-1}) + \beta (1 - \alpha) \frac{\bar{Y}}{K} \rho_z z_t$$
$$+ \beta \phi \mathbb{E}[\dot{k}_{t+2} - \dot{k}_{t+1}] + \phi \dot{k}_t - (\alpha \delta + \phi) \dot{k}_{t+1}$$

(20)

We use the conjecture for $\dot{k}$

$$\mathbb{E} \left[ \dot{k}_{t+2} \right] = D_{k,b} b_t X_t + D_{k,k} D_k X_t + \rho_z D_{k,z} z_t + ((1 + \rho_g) D_{k,\zeta_1} + D_{k,\zeta_2}) \zeta_t - \rho_g D_{k,\zeta_1} \zeta_{t-1}$$
$$\mathbb{E} \left[ \dot{k}_{t+1} \right] = D_{k,b} b_t + D_{k,k} \dot{k}_t + D_{k,z} z_t + D_{k,\zeta_1} \zeta_t + D_{k,\zeta_2} \zeta_{t-1}$$

to collect the terms for $\zeta_t$ from (20):

$$\left( \beta \phi \rho_g + \beta (1 - \alpha) \frac{\bar{Y}}{K} \alpha \rho_g - \phi \right) - (\alpha \delta + \phi) D_{k,\zeta_1} + \beta \phi (D_{k,b} b_{\zeta_1} + D_{k,k} D_{k,\zeta_1} + \rho_g D_{k,\zeta_1} + D_{k,\zeta_2}) = 0$$

(21)

Similarly, collecting the terms for $\zeta_{t-1}$:

$$- \left( \beta \phi \rho_g + \beta (1 - \alpha) \frac{\bar{Y}}{K} \alpha \rho_g - \phi \right) - (\alpha \delta + \phi) D_{k,\zeta_2} + \beta \phi (D_{k,b} b_{\zeta_2} + D_{k,k} D_{k,\zeta_2} - \rho_g D_{k,\zeta_1} - D_{k,\zeta_2}) = 0$$

(22)

Combining (21) and (22), as $\bar{Q} \to 1$, and consequently $\beta \to 1$, we have

$$(\alpha \delta + \phi) (D_{k,\zeta_1} + D_{k,\zeta_2}) - \phi D_{k,k} (D_{k,\zeta_1} + D_{k,\zeta_2}) = 0$$

which leads to

$$(D_{k,\zeta_1} + D_{k,\zeta_2})(\alpha \delta + \phi (1 - D_{k,k})) = 0$$

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As $\bar{Q} \to 1$ and $\psi \to 0$, $D_{k,k} \to D^*_{k,k} < 1$, therefore

$$D_{k,\zeta_1} + D_{k,\zeta_2} = 0$$
References


