Household (h,f)’s Utility in period t –

(1) \[ U_t(h,f) = E_t \sum_{j=t}^{\infty} \beta^{j-t} [u(C_j(h,f)) - g(N_{s,t}(h,f)) + v(M_j(h,f)/P_j)] \]

where \( C_j(h,f) \) is household consumption of the composite consumption good;
when we want specific functional forms, we will specify “constant elasticity” utility:
\[ u(C) = (1-\gamma)^{-1} C^{-\gamma}, \quad g(N) = A(1+\chi)^{-1} N^{1+\chi} \quad \text{and} \quad v(M/P) = V(1-\nu)^{-1} (M/P)^{1-\nu}. \]

Bundlers –
Bundler of the composite consumption good C –

(2) \[ C_{s,t} = \prod_{s=1}^{S} C_{s,t}^{1/S} \quad \text{Consumption Good } C_{s,t} \]

\[ P_{s,t} = S \prod_{s=1}^{S} P_{s,t}^{1/S} \quad \text{Price (of bundler) for the Consumption Good} \]

\[ (1/S)P_{s,t}C_{s,t} = P_{s,t}C_{s,t} \quad \text{Demand (of bundler) for the Sectoral Goods} \]

Bundler of Sectoral good Y_{s,t} –

(3) \[ Y_{s,t} = \int_s^{s+1} Y_{s,t}(f)^{(1-\Theta)/\Theta} df \quad \text{Sectoral Good } Y_{s,t} \]

\[ P_{s,t} = \int_s^{s+1} P_{s,t}(f)^{1-0 df}^{1/(1-\Theta)} \quad \text{Price (of bundler) for the sectoral good } Y_{s,t} \]

\[ Y_{s,t}(f) = (P_{s,t}/P_{s,t}(f))^{\phi} Y_{s,t} \quad \text{Demand (of the bundler) for good of firm f} \]

Bundler of Firm f’s labor input –

(4) \[ N_{s,t}(f) = \int_0^{\phi} N_{s,t}(h,f)^{\phi-1} dh \quad \text{Labor input for } Y_{s,t}(f) \]

\[ W_{s,t}(f) = \int_0^{1} W_{s,t}(h,f)^{1-\phi} dh \quad \text{Price (of bundler) for the labor input for } Y_{s,t}(f) \]

\[ N_{s,t}^d(h,f) = (W_{s,t}(f)/W_{s,t}(h,f))^{\phi} N_{s,t}(f) \quad \text{Demand (of the bundler) for labor of household (h,f)} \]
The algebra of competitive bundlers:

Here, we assume Chari, Kehoe and McGrattan’s (2000) “bundlers”:

The “bundler” for sector $s$ is a competitive (or zero profit) agent who buys the firms’ $Y_{s,t}(f)$ at the price $P_{s,t}(f)$, bundles them into the sectoral good $Y_{s,t} = \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{\theta/(\theta-1)}$, and sells it at the price $P_{s,t}$.

The bundler minimizes the cost of producing a given amount of $Y_{s,t}$:

$$\min_{Y_{s,t}(f)} \int_0^1 P_{s,t}(f) Y_{s,t}(f) \, df \quad \text{s.t.} \quad Y_{s,t} = \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{\theta/(\theta-1)}$$

$$\mathcal{L} = \int_0^1 P_{s,t}(f) Y_{s,t}(f) \, df + \mu \left\{ Y_{s,t} - \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{\theta/(\theta-1)} \right\}$$

Note: Lagrangian multiplier $\mu = MC = P_{s,t}$, since bundler is competitive

First Order Condition –

$$P_{s,t}(f) = P_{s,t} \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{\theta/(\theta-1)} \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{1/(\theta-1)} = P_{s,t} Y_{s,t}^{1/(\theta-1)} Y_{s,t}(f)^{\theta/(\theta-1)}$$

To find $P_{s,t}$, use FOC to eliminate $Y_{s,t}(f)$ in $Y_{s,t} = \left[ \int_0^1 Y_{s,t}(f) \theta \, df \right]^{\theta/(\theta-1)}$

$$Y_{s,t} = \left[ \int_0^1 (P_{s,t}(f)/P_{s,t})^\theta Y_{s,t} \theta \, df \right]^{\theta/(\theta-1)}$$

$$P_{s,t}^\theta = \left[ \int_0^1 [P_{s,t}(f)]^\theta \, df \right]^{\theta/(\theta-1)} \Rightarrow P_{s,t} = \left[ \int_0^1 [P_{s,t}(f)]^\theta \, df \right]^{1/(1-\theta)}$$

Collecting results:

$$P_{s,t} = \left[ \int_0^1 [P_{s,t}(f)]^\theta \, df \right]^{1/(1-\theta)} \quad \text{Price of sectoral good } Y_{s,t}$$

$$Y^d_{s,t}(f) = (P_{s,t}/P_{s,t}(f))^\theta Y_s \quad \text{Demand for good output of firm } f$$
Budget Constraint of Household \((h,f)\) –

\[ M_{t+1}(h,f) + E_t[\delta_{t+1} B_{t+1}(h,f)] + P_t C_t(h,f) + P_t T_c = W_{s,t}(h,f)N_{s,t}(h,f) + M_{t+1}(h,f) + B_t(h,f) + \Omega_t(h,f) \]

where \(B_{t+1}\) is a portfolio of state-contingent claims, \(\delta_{t+1}\) is the stochastic discount factor, \(E\) is an expectation over states; \(\Omega\) are dividends, and \(T_c\) is a lump-sum tax (or transfer).

Household-\((h,f)\)’s Intertemporal Maximization Problem –

Max (1), s.t. the Budget Constraint, (5), and labor demand, (4):

\[ \mathcal{Q} = E_t \sum_{t=0}^{\infty} \beta^t \{ [u(C_t(h,f)) - g(N_{s,t}(h,f)) + v(M_t(h,f)/P_t)] \]

\[ + \lambda_t[W_{s,t}(h,f)N_{s,t}(h,f) + M_{t+1}(h,f) + B_t(h,f) + \Omega_t(h,f) - M_t(h,f) - E_t[\delta_{t+1} B_{t+1}(h,f)] - P_t C_t(h,f) - P_t T_c] \}

FOC that hold independent of the type of nominal inertia –

\[ C_t: \lambda_t(h,f)P_t = u'(\cdot) \quad \text{(or with log utility)} \quad \lambda_t(h,f)P_t = 1/C_t(h,f) \]

\[ B_{t+1}: \delta_{t+1} = \beta \lambda_{t+1}(h,f)/\lambda_t(h,f) \]

\[ M_t: \nu'(\cdot) = \lambda_t(h,f) [1 - \beta E_t[\lambda_{t+1}(h,f)/\lambda_t(h,f)] = \lambda_t(h,f) [1 - E_t(\delta_{t+1})] \]

\[ M_t: \nu'(\cdot) = \lambda_t(h,f) [1 - \beta E_t(\delta_{t+1})] \quad \text{(or with log utility)} \quad M_t(h,f) = \nu_t P_t C_t(h,f)/[1 - E_t(\delta_{t+1})] \]

Remarks (on complete contingent claims modeling):

1. Cochrane (2001, Ch. 3) provides a good (and brief) discussion of complete contingent claims markets. Our parsimonious notation follows Woodford (1997).

2. Following Cochrane, \(p(\text{portfolio}) = \sum_\sigma p(\sigma)B(\sigma) = \sum_\sigma \pi(\sigma)[p(\sigma)/\pi(\sigma)]B(\sigma) = \sum_\sigma \pi(\sigma)\delta(\sigma)B(\sigma) = E[\delta(\sigma)B(\sigma)]\), where \(\sigma\) is the state of nature, \(p(\sigma)\) is the period \(t\) price of a dollar in state \(\sigma\) in period \(t+1\), \(B(\sigma)\) is the number of claims in the portfolio, and \(p(\text{portfolio})\) is the price of the portfolio. The “stochastic discount factor” \(\delta(\sigma)\) is the price of a dollar in state \(\sigma\) divided by the probability of state \(\sigma\) occurring. \(B_{t+1}(h,f)\) is \(B(\sigma)\). (6), like (7), when forwarded, has to hold for each state. Note: all households face the same \(\delta(\sigma)\). So, (6) & (7) imply that all households have the same actual rate of growth of nominal income.

3. The “risk free” rate of return –

Consider a bond that costs 1 dollar in \(t\) and pays 1 dollars in all states in \(t+1\).

\[ 1 = E_t[\delta_{t+1} 1_t] = 1_t \cdot E_t[\delta_{t+1} 1_t] = \beta E_t[\lambda_{t+1} / \lambda_t] = \beta E_t[P_t C_t(h,f)/P_{t+1} C_{t+1}(h,f)] \text{ for all } (h,f) \]
Using labor demand curve (4), the Household Intertemporal Maximization Problem becomes:

\[
\text{Max } \sum_{t=1}^{\infty} \beta^{t-1} \left\{ u(C_t(h,f)) - g(W_{s,t}(h,f)/W_{s,t}(f)) N_{s,t}(f) \right\} + v(M_{c,t}(h,f)/P_c) \]

\[+ \lambda_c [W_{s,t}(h,f)][(W_{s,t}(h,f)/W_{s,t}(f))]^\phi N_{s,t}(f)] + M_{c,t}(h,f) + B_{c}(h,f) + \Omega(t,h,f) - \text{[as above]} \]

If (h,f) works at a flexible-wage firm:

\[
W_{s,t}(h,f): \phi g'(\cdot) W_{s,t}(h,f)^{1-\phi}(1/W_{s,t}(f))^{\phi} N_{s,t}(f) + \lambda_c (1-\phi) W_{s,t}(h,f)^{\phi}(1/W_{s,t}(f))^{\phi} N_{s,t}(f) = 0
\]

\[
\phi g'(\cdot) W_{s,t}(h,f)^{1-\phi}(1/W_{s,t}(f))^{\phi} N_{s,t}(f) = (\phi-1)[u'(\cdot)/P_c] W_{s,t}(h,f)^{\phi}(1/W_{s,t}(f))^{\phi} N_{s,t}(f)
\]

\[
\mu_w g'(\cdot) N_{s,t}(h,f) W_{s,t}(h,f)^{1-\phi} = [u'(\cdot)/P_c] N_{s,t}(h,f) \text{ where } \mu_w = \phi/(\phi-1) > 1
\]

canceling the \(N_{s,t}(f)\) (which can’t be done in the sticky wage case)

\[
(9)_{\text{flex}} \quad W_{s,t}(h,f)/P_c = \mu_w [g'(\cdot)/u'(\cdot)] = \mu_w A N_{s,t}(h,f)^2 C_t(h,f)
\]

Remarks:

1. The interpretation of \((9)_{\text{flex}}\):

   Can be written as: \((9)' \quad g'(\cdot) = (1/\mu_w)[W(h,f)/P]u'(\cdot)

   The LHS of \((9)'\) is the disutility of working one more “hour”.

2. The distortion created by monopolistic competition:

   When monopolistic wage setters increase work, the wage bill goes up less than 1 for 1 since the \(W(h,f)\) falls. \(\mu = \phi/(\phi-1) > 1\) is the distortion (or markup) created by monopolistic competition. It makes households work too little. As \(\phi \to \infty\) (and \(\mu \to 1\)), the demand curves become infinitely elastic, and the distortion is eliminated, leaving the private marginal benefit of work equal to the marginal cost of work.

3. As G&K (2001) have noted with respect to price markups, there is an analogy between the \(1/\mu\) and the income tax; either leads to too little work effort. More later.
If \((h,f)\) works at a fixed-wage firm:

\[
\mu_w E_{t-1}[g'(\cdot)(N_{s,t}(h,f)/W_{s,t}(h,f)) = E_{t-1}[(u'(\cdot)/P_t)N_{s,t}(h,f)] \quad \text{or}
\]

\( W_{s,t}(h,f) = \mu_w E_{t-1}[g'(\cdot)(N_{s,t}(h,f)/E_{t-1}[(u'(\cdot)/P_t)N_{s,t}(h,f)] \quad \text{or}
\]

\[
= \mu_w E_{t-1}[A_tN_{s,t}(h,f)^{1+\gamma}]/E_{t-1}[(1/P_tC_t(h,f))N_{s,t}(h,f)]
\]

**Firms:**

The present value of firm-\(f\)’s nominal profits stream \(\{R_{s,f}\}\) is:

\[
PV_t(f) = E_\sum_{t=0}^\infty \delta_t R_{s,f},
\]

Firms maximize their market value, \(PV\); multiplying by \(\lambda_t\) and using (6), we can write their objective function as:

\[
(10) \quad MV_t = \lambda_t PV_t(f) = E_\sum_{t=0}^\infty \beta^{t}\lambda_t R_{s,f}
\]

Recall:

Firm-\(f\)’s Production Function: \(Y_{s,t}(f) = Z_{s,t}N_{s,t}(f)\)

Demand (of the bundler) for Firm-\(f\)’s product: \(Y_{s,t}^d(f) = (P_{s,t}(f)/P_{s,t})^{\theta}Y_{s,t}\)

Firm-\(f\) chooses \(P_t(f)\) to max its \(MV_t\):

\[
MV_t = E_\sum_{t=0}^\infty \beta^{t}\lambda_t R_{s,f} = E_\sum_{t=0}^\infty \beta^{t}\lambda_t [P_{s,t}(f)Y_{s,t}(f) - W_{s,t}(f)Y_{s,t}(f)/Z_{s,t}]
\]

\[
= E_\sum_{t=0}^\infty \beta^{t}\lambda_t [P_{s,t}(f)(P_{s,t}(f)/P_{s,t})^{\theta}Y_{s,t} - (W_{s,t}(f)/Z_{s,t})(P_{s,t}(f)/P_{s,t})^{\theta}Y_{s,t}]
\]

\[
= E_\sum_{t=0}^\infty \beta^{t}\lambda_t Y_{s,t} [P_{s,t}(f)^{1-\theta}(1/P_{s,t})^{\theta} - (W(f)/Z_{s,t})(P_{s,t}(f)^{1-\theta}(1/P_{s,t})^{\theta})]
\]

if \(f\) is a flexible price firm (letting \(\mu_p = \theta/(\theta-1) > 1\):

\[
P_{s,t}(f) = \lambda_t Y_{s,t}[(1-\theta)P_{s,t}(f)^{\theta}/(1/P_{s,t})^{\theta} + \theta(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{1-\theta}(1/P_{s,t})^{\theta}] = 0
\]

\[
\lambda_t Y_{s,t}(f) = \mu_p \lambda_t Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}
\]

canceling the \(\lambda_t Y_{s,t}(f)\) (which we can't do in the fixed price case)

\[
(11)_{\text{flex}} \quad P_{s,t}(f) = \mu_p [W_{s,t}(f)/Z_{s,t}]
\]
If \( f \) is a fixed price firm (letting \( \mu_p = \theta/(\theta-1) > 1 \):

\[
P_{s,t}(f) = \mu_p E_{t-1}[(u'()\mu_p)Y_{s,t}(f)W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}]
\]

\[
E_{t-1}[u'()P_p)Y_{s,t}(f)] = \mu_p E_{t-1}[u'()P_p)Y_{s,t}(f)(W_{s,t}(f)/Z_{s,t})P_{s,t}(f)^{-1}]
\]

\[(11)_{\text{fixed}} P_{s,t}(f) = \mu_p E_{t-1}[(u'()P_p)Y_{s,t}(f)W_{s,t}(f)/Z_{s,t})]/E_{t-1}[u'()P_p)Y_{s,t}(f)]
\]

\[
= \mu_p E_{t-1}[(1/P\mu_pC_t(h,f))Y_{s,t}(f)W_{s,t}(f)/Z_{s,t})]/E_{t-1}[(1/P\mu_pC_t(h,f))Y_{s,t}(f)]
\]

(with log utility)

Remark:

If the firm is setting \( P_{s,t}(f) \) for multiple periods – as in Taylor or Calvo contracts – we get an obvious generalization of \((16)_{\text{fixed}}\); see CCD (2002b), The “New-Keynesian” Phillips Curve.

A Fundamental Relationship between Work and Leisure:

Note: \((9)_{\text{fixed}}, (11)_{\text{fixed}}\) and \((2c)\) ⇒

\[
P_{s,t} = \mu_p E_{t-1}[(u'()P_p)Y_{s,t}(W_{s,t}/Z_{s,t})]/E_{t-1}[(u'()P_p)Y_{s,t}]
\]

\[
\Rightarrow E_{t-1}[(u'()P_p)P_{s,t}Y_{s,t}] = \mu_p E_{t-1}[(u'()P_p)W_{s,t}N_{s,t}]
\]

\[
W_{s,t} = \mu_w E_{t-1}[g'()N_{s,t}]/E_{t-1}[(u'()P_p)N_{s,t}]
\]

\[
\Rightarrow E_{t-1}[(u'()P_p)W_{s,t}N_{s,t}] = \mu_w E_{t-1}[g'()N_{s,t}]
\]

\[
So, E_{t-1}[(u'()P_p)P_{s,t}Y_{s,t}] = \mu_p E_{t-1}[g'()N_{s,t}]
\]

Using constant expenditure shares: \(\mu E_{t-1}[g'()N_{s,t}] = E_{t-1}[(u'()P_p)P_{s,t}Y_{s,t}] = E_{t-1}[(u'()P_p)P_sC_t/S]\)

Or finally: \(\mu E_{t-1}[g'()N_{s,t}] = E_{t-1}[(u'()C_s(h,f)]\)

Thus we have:

**Lemma 1:** Relationship between the (expected) utility of consumption and disutility of work –

(A) \((16)_{\text{fixed}}, (17)_{\text{fixed}}\) and \((2c)\) ⇒ \(E_{t-1}[(u'()C_t(h,f)] = \mu E_{t-1}[g'()N_{s,t}]\)

(B) \(u(C) = (1-\gamma)^{-1}C^\gamma\) and \(g(N) = A_t(1+\chi)^{-1}N^{(1+\gamma)}\) ⇒ \(E_{t-1}[u(\gamma)] = [(1+\chi)/(1-\gamma)]\mu E_{t-1}[g(\gamma)]\)

(C) \(u(C) = \log(C)\) and \(g(N) = A_t(1+\chi)^{-1}N^{(1+\gamma)}\) ⇒ \(E_{t-1}[g(\gamma)] = [1/\mu(1+\gamma)]\)
Since the first order conditions for households and firms in a given sector are identical, we look for an equilibrium in which:

\[ N_{s,t}(f) = \left[ \int_0^{N_{s,t}(h,f)} (\phi f/e\, dh) \right]^{e/(\phi - 1)} = N_{s,t}(h,f) \]

\[ W_{s,t}(f) = \left[ \int_0^{W_{s,t}(h,f)} f^i (\phi f/e\, dh) \right]^{e/(i - 1)} = W_{s,t}(h,f) \]

So, we can drop the indices (f) and the (h,f), and let:

\[ N_{s,t}(h,f) = N_{s,t}(f) \]

\[ W_{s,t}(h,f) = W_{s,t}(f) \]

Similarly:

\[ Y_{s,t}(f) = Y_{s,t} \]

\[ P_{s,t}(f) = P_{s,t} \]

Aggregation: (Recall, there are S sectors, and measure S households)

\[ \int_0^1 \int_0^{N_{s,t}(h,f)} \frac{C_{s,t}(h,f) dh}{df} = \frac{S C_{s,t}(h,f)}{S C_{s,t}} = Y_{s,t}(f) = \left[ \int_s^{s+1} Y_{s,t}(f)^{(0.1 - 0.1)} df \right]^{0.1} \]

\[ \int_0^1 \int_0^{W_{s,t}(h,f)} \frac{C_{s,t}(h,f) dh}{df} = \frac{S C_{s,t}(h,f)}{S C_{s,t}} = Y_{s,t}(f) = \left[ \int_s^{s+1} Y_{s,t}(f)^{(0.1 - 0.1)} df \right]^{0.1} \]

An Equilibrium is characterized by –

In equilibrium, the wage and price equations become (for \( s = 1, 2, \ldots, S \)):

\[ (12) \text{ flex } W_{s,t}/P_t = \mu_w[g'(N_{s,t})/u'(C_t)] \]

\[ (12) \text{ fixed } W_{s,t} = \mu_w E_{t+1}[g'(N_{s,t})N_{s,t}]/E_{t+1}[(u'(C_t)/P_t)N_{s,t}] \]

\[ (13) \text{ flex } P_{s,t} = \mu_p[W_{s,t}/Z_{s,t}] \]

\[ (13) \text{ fixed } P_{s,t} = \mu_p E_{t+1}[(u'(C_t)/P_t)Y_{s,t}(W_{s,t}/Z_{s,t})]/E_{t+1}[(u'(C_t)/P_t)Y_{s,t}] \]

The other equilibrium conditions become:

\[ (14) \text{ } P_t C_t = S P_{s,t} C_{s,t} \]

\[ (15) \text{ } P_t = S \Pi_{s=1}^S P_{s,t}^{1/S} \]

\[ (16) \text{ } I_t^{-1} = E_t[\delta_{t+1}] = \beta E_t[(u'(C_{t+1})/u'(C_t))(P_t/P_{t+1})] \]

\[ (17) \text{ } V_t' = (M_t/P_t) = u'(C_t)[1 - I_t^{-1}] \]

Monetary policy procedures:

\[ (18) \text{ } P_t C_t = \Omega(A_t, Z_{1,t}, \ldots, Z_{S,t}) \]
The flexible wage/price “benchmark” –

(12)flex and (14) \[ W_{s,t} = \mu_{w}A_{N,s}^{\chi}P_{s,t} \]
\[ \Rightarrow \frac{W_{s,t}}{P_{s,t}} = \frac{\mu_{w}A_{N,s}^{\chi}}{P_{s,t}} \]
(13)flex \[ W_{s,t}/P_{s,t} = (1/\mu_{p})Z_{s,t} = N_{s,t} = \mu^{-1(1+\chi)}A_{t}^{-1(1+\chi)} \]
where \( \mu = \mu_{p}\mu_{w} \) for all \( s \) \( \Rightarrow \) wages equalize
(18) \[ = W_{s,t} = \mu_{w}A_{N,s}^{\chi}P_{s,t} = \mu_{w}A_{t}^{\chi}N_{s,t}^{\chi}Q_{t} = \mu_{w}^{\chi(1+\chi)}A_{t}^{1(1+\chi)}Q_{t} \]
for all \( s \)

And for any two sectors \( s \) and \( s' \), (13)flex \[ \Rightarrow \]
\[ \frac{P_{s,t}}{P_{s',t}} = \left( \frac{W_{s,t}}{Z_{s,t}} \right) \frac{\mu_{p}[W_{s,t}/Z_{s,t}]}{\mu_{p}[W_{s',t}/Z_{s',t}]} = Z_{s,t}/Z_{s',t} \]
\[ \frac{P_{s',t}}{P_{t}} = SZ_{s',t}^{-1} \Pi_{s=1}^{S}Z_{s,t}^{1/S} \]
and the aggregate price level becomes:
\[ P_{t}C_{t} = \Omega_{t} = \Omega_{t}/C_{t} = \Omega_{t}/\Pi_{s=1}^{S}C_{s,t}^{1/S} \]

Lemma 2: The Flexible Wage/Price Solution –

If \( u(C_{t}) = \log(C_{t}) \) and \( g(N_{t}) = At(1+P)^{-1}N_{t}^{1+P} \), then
(A) \[ P_{s,t} = \mu_{p}(W_{s,t}/Z_{s,t}) \]
(B) \[ N_{s,t} = \mu^{-1(1+\chi)}A_{t}^{-1(1+\chi)} \]
and
(C) \[ Y_{s,t} = Z_{s,t}N_{s,t} = \mu^{-1(1+\chi)}A_{t}^{-1(1+\chi)}Z_{s,t} = SC_{s,t} \]
(D) \[ C_{s,t}/C_{s,t} = P_{s,t}/P_{s,t} = Z_{s,t}/Z_{s',t} \text{ and } P_{s,t}/P_{s,t} = (\Pi_{s=1}^{S}Z_{s,t}^{1/S})/SZ_{s',t} \]
(E) \[ C_{t} = (1/S)\mu^{-1(1+\chi)}A_{t}^{-1(1+\chi)} \Pi_{s=1}^{S}Z_{s,t}^{1/S} \]
(F) \[ P_{t} = \Omega_{t}S^{1(1+\chi)}A_{t}^{1(1+\chi)}S^{1/S}Z_{s,t}^{1/S} \text{ and } W_{s,t} = \mu_{w}^{\chi(1+\chi)}A_{t}^{1(1+\chi)}Q_{t} \]
where *'s denote flexible wage/price values and \( \mu = \mu_{p}\mu_{w} \) is the combined markup.

Derivation of labor supply and demand for sector \( s \):

Labor demand:
\[ W_{s,t}/P_{t} = \mu_{w}[g'(\cdot)/u'(\cdot)] = \mu_{w}[A_{N,s}^{\chi}]C_{t}^{\gamma} = \mu_{w}[A_{N,s}^{\chi}](C_{s,t}(SP_{s,t}/P_{t}))^{\gamma} \]
\[ (W_{s,t}/P_{t})(P_{t}/P_{s,t})^{\gamma} = \mu_{w}[A_{N,s}^{\chi}](Y_{s,t})^{TS^{-1}} = \mu_{w}[A_{N,s}^{\chi}](N_{s,t}Z_{s,t})^{YS^{-1}} = \mu_{w}[A_{N,s}^{\chi}](N_{s,t}Z_{s,t})^{YS^{-1}} \]
\[ (W_{s,t}/P_{t})(P_{s,t}/P_{t})^{\gamma} = (W_{s,t}/P_{s,t})(P_{s,t}/P_{t})^{\gamma-1} = \mu_{w}[A_{N,s}^{\chi}](Z_{s,t}^{Y}S_{s,t}^{Y-1}) \]
Labor Supply:
\[ W_{s,t}/P_{s,t} = \mu_{w}A_{N,s}^{\chi}(Y_{s,t})^{Y+Z_{s,t}^{Y+1}}(P_{s,t}/P_{s,t})^{Y-1}S_{s,t}^{Y-1} \]
Taking logs of each curve (and letting small letters denote logs), we have:
\[ w_{s,t} - p_{s,t} = \log(1/\mu_{p}) + z_{s,t} \]
\[ w_{s,t} - p_{s,t} = \text{constant} + (\gamma+\chi)n_{s,t} + a_{t} + \{\gamma + (\gamma-1)(1/S) - 1\}z_{s,t} + (1-\gamma)(1/S)\sum_{s=1}^{S}z_{s,t} \]
= \text{constant} + (\gamma+\chi)n_{s,t} + a_{t} + [1 + (\gamma-1)(1/S)]z_{s,t} + (1-\gamma)(1/S)\sum_{s=1}^{S}z_{s,t}
Solution for sectors with sticky prices (and sticky or flexible wages):

Fixed expenditure shares ⇒ $\Omega_t = P_{s,t}Y_{s,t}$

Lemma 1 ⇒ $E_{t-1}[g(\cdot)] = [1/\mu(1+\chi)] ⇒ E_{t-1}[A_tN_{s,t}^\Psi] = (1/\mu)$ where $\Psi = 1+\chi$

⇒ $E_{t-1}[A_t(Y_{s,t}/Z_{s,t})^\Psi] = (1/\mu) ⇒ E_{t-1}[A_t(\Omega_t/P_{s,t}Z_{s,t})^\Psi] = 1/\mu$

⇒ $P_{s,t} = \mu^{1/\Psi}\{E_{t-1}[A_t(\Omega_t/Z_{s,t})^\Psi]\}^{1/\Psi}$

Recall: $Y_{s,t}^* = \mu^{-1/\Psi}A_t^{-1/\Psi}Z_{s,t}$

$P_{s,t}^* = \Omega_t/Y_{s,t}^* = \Omega_t\mu^{1/\Psi}A_t^{1/\Psi}/Z_{s,t}$

Assuming log-normality:

$p_{s,t} = (1/\Psi)\log(\mu) + (1/\Psi)\log E_{t-1}[A_t(\Omega_t/Z_{s,t})^\Psi]
= (1/\Psi)\log(\mu) + (1/\Psi)E_{t-1}[a_t + \Psi(\omega_t - z_{s,t})] + 1/2(1/\Psi)\text{VAR}_{t-1}[a_t + \Psi(\omega_t - z_{s,t})]
= (1/\Psi)\log(\mu) + E_{t-1}[\omega_t - z_{s,t} + a_t/\Psi] + 1/2\Psi\text{VAR}_{t-1}[\omega_t - z_{s,t} + a_t/\Psi]
= p_{s,t}^* - \{\omega_t - E_{t-1}[\omega_t]\} + \{z_{s,t} - E_{t-1}[z_{s,t}]\} - \{(a_t/\Psi) - E_{t-1}[a_t/\Psi]\} + 1/2\Psi\text{VAR}_{t-1}[\omega_t - z_{s,t} + a_t/\Psi]

y_{s,t} - y_{s,t}^* = - (p_{s,t} - p_{s,t}^*)
= \{\omega_t - E_{t-1}[\omega_t]\} - \{z_{s,t} - E_{t-1}[z_{s,t}]\} + \{(a_t/\Psi) - E_{t-1}[a_t/\Psi]\} - 1/2\Psi\text{VAR}_{t-1}[\omega_t - z_{s,t} + a_t/\Psi]
Solution for sectors with sticky wages and flexible prices:

Fixed expenditure shares ⇒ \( \Omega_t = P_{s,t} * Y_{s,t} = \mu_p [W_{s,t}/Z_{s,t}] Y_{s,t} = \mu_p W_{s,t} N_{s,t} \)

Lemma 1 ⇒ \( E_{t-1} \{ g(\cdot) \} = [1/\mu(1+\chi)] \) ⇒ \( E_{t-1} [A_t N_{s,t}^\Psi] = 1/\mu \) where \( \Psi = 1+\chi \)

⇒ \( E_{t-1} [A_t N_{s,t}^\Psi] = (1/\mu_p)^\Psi E_{t-1} [A_t (\Omega_t/W_{s,t})^\Psi] = 1/\mu \)

⇒ \( W_{s,t} = (1/\mu_p) \{ E_{t-1} [A_t \Omega_t^\Psi] \}^{1/\Psi} \)

Recall: \( Y_{s,t}^\Psi = -1/\Psi \) \( A_{t} \) \( N_{s,t}^\Psi = (1/\mu_p)^{1/\Psi} \) \( A_t \)

Assuming log-normality:

\[
 w_{s,t} = \log(\mu_{1/\Psi}/\mu_o) + (1/\Psi) \log E_{t-1} [A_t \Omega_t^\Psi] \\
= \log(\mu_{1/\Psi}/\mu_o) + (1/\Psi) E_{t-1} [\psi \omega_i + a_i] + \frac{1}{2}(1/\Psi) \text{VAR}_{t-1} [\psi \omega_i + a_i] \\
= \log(\mu_{1/\Psi}/\mu_o) + E_{t-1} [\omega_i + a_i/\Psi] + \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i + a_i/\Psi] \\
= w_{s,t}^* - \{ \omega_{i} - E_{t-1} [\omega_{i}] \} - \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} + \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i + a_i/\Psi] \\
\]

\[
 n_{s,t} - n_{s,t}^* = - (w_{s,t} - w_{s,t}^*) = \{ \omega_{i} - E_{t-1} [\omega_{i}] \} + \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} - \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i + a_i/\Psi] \\
\]

Lemma 3: Sticky wage/price solutions –

In sectors where wages are sticky and prices are flexible:

A. \( y_{s,t} - y_{s,t}^* = \{ \omega_{i} - E_{t-1} [\omega_{i}] \} + \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} - \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i + a_i/\Psi] = n_{s,t} - n_{s,t}^* \)

B. \( w_{s,t} = w_{s,t}^* - \{ \omega_{i} - E_{t-1} [\omega_{i}] \} - \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} + \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i + a_i/\Psi] \)

In sectors where prices are sticky and wages are either flexible or sticky:

C. \( y_{s,t} - y_{s,t}^* = \{ \omega_{i} - E_{t-1} [\omega_{i}] \} - \{ z_{s,t} - E_{t-1} [z_{s,t}] \} + \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} - \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i - z_{s,t} + a_i/\Psi] \)

D. \( p_{s,t} = p_{s,t}^* - \{ \omega_{i} - E_{t-1} [\omega_{i}] \} + \{ z_{s,t} - E_{t-1} [z_{s,t}] \} - \{ (a_i/\Psi) - E_{t-1} [a_i/\Psi] \} + \frac{1}{2} \psi \text{VAR}_{t-1} [\omega_i - z_{s,t} + a_i/\Psi] \)

where \( y_{s,t}^* \) are flex-price levels of output, and \( w_{s,t}^* \) and \( p_{s,t}^* \) are “notional” prices (or “shadow” prices) of labor and output; \( \Psi = 1+\chi \).
We assume the goal of monetary policy is to maximize average household utility:

(19) \( W_t = E_{t+1} \sum_{t=0}^{\infty} \beta^t [u(C_t) - (1/S) \sum_{s=1}^{S} g(N_{s,t})] \)

**Proposition 1:** Let \( u(C) = \log(C) \) and \( g(N) = \psi^1 AN^\psi \); let \( Z_{s,t} \) have a log-normal distribution; let \( W \) be the set of sectors that have fixed wages and flexible prices, and let \( P \) be the set of sectors that have sticky prices and flexible or sticky wages; and let \( g_{s,t} = y_{s,t} - y_{s,t}^* \) be the “output gap” in sector \( s \). Then, the goal of monetary policy is to choose a rule, \( \omega(\cdot) \), that maximizes \( \sum_{s=1}^{S} E_{t+1} c_{s,t} \); moreover, \( \sum_{s=1}^{S} E_{t+1} c_{s,t} \) can be expressed in three ways:

(A) \( \sum_{s=1}^{S} E_{t+1} c_{s,t} = -S \log(S) + \sum_{s=1}^{S} E_{t+1} y_{s,t}^* - 1/2 \sum_{s \in W} \text{VAR}_t \{\omega_t + a_t/\psi\} \)

(B) \( \sum_{s=1}^{S} E_{t+1} c_{s,t} = -S \log(S) + \sum_{s=1}^{S} E_{t+1} y_{s,t}^* - 1/2 \sum_{s \in W} \text{VAR}_t \{g_{s,t}\} - 1/2 \sum_{s \in P} \text{VAR}_t \{g_{s,t}\} \)

(C) \( \sum_{s=1}^{S} E_{t+1} c_{s,t} = -S \log(S) + \sum_{s=1}^{S} E_{t+1} y_{s,t}^* - 1/2 \sum_{s \in W} \text{VAR}_t \{w_{s,t}^*\} - 1/2 \sum_{s \in P} \text{VAR}_t \{p_{s,t}^*\} \)

**Proof:**

Lemma 1 \( \Rightarrow \) \( W_t = E_{t+1} \sum_{t=0}^{\infty} \beta^t [\log(C_t) - (1/S) \sum_{s=1}^{S} g(N_{s,t})] \) and the solution and the maximization are static; since monetary policy can not affect the expected disutility of work, the goal reduces to maximizing \( E_{t+1} \log(C_t) \), which by the fixed expenditure shares is proportional to \( \sum_{s=1}^{S} E_{t+1} c_{s,t} \). The expressions for \( \sum_{s=1}^{S} E_{t+1} c_{s,t} \) follow directly from Lemma 3.

For convenience, we re-write Lemma 2 in log terms:

**Lemma 2a:** The benchmark flexible wage/price solution is:

(A) \( p_{s,t}^* = \log(\mu_p) + w_{s,t}^* - z_{s,t} \)

(B) \( n_{s,t}^* = -\psi^{-1} \log(\mu) - \psi^{-1} a_t \) and \( w_{s,t}^* = \log(\mu_w) - \chi \psi^{-1} \log(\mu) + \psi^{-1} a_t + \omega_t \) (for all \( s \))

(C) \( y_{s,t} = z_{s,t} + n_{s,t}^* = -\psi^{-1} \log(\mu) - \psi^{-1} a_t + z_{s,t} = c_{s,t}^* + \log(S) \)

(D) \( p_{s,t}^* - p_{s,t}^* = z_{s,t} - z_{s,t}^* \) and \( p_{s,t}^* - p_{t}^* = -\log(S) - z_{s,t} + (1/S) \sum_{s=1}^{S} z_{s,t} \)

(E) \( c_{s,t}^* - c_{s,t}^* = z_{s,t} - z_{s,t}^* \)

(F) \( c_{t}^* = -\psi^{-1} \log(\mu) - \psi^{-1} a_t + (1/S) \sum_{s=1}^{S} z_{s,t} - \log(S) \) and

\( p_{t}^* = \log(S) + \psi^{-1} \log(\mu) + \psi^{-1} a_t - (1/S) \sum_{s=1}^{S} z_{s,t} \)

where \( \mu = \mu_p \mu_w \) is the combined markup.
Inefficiency:

Efficiency requires that the marginal product of labor (MPL) in any sector, s, be equal to the marginal rate of substitution (MRS) between leisure and the good produced in sector s. Dropping time subscripts, the efficiency condition is:

\[
MRS = \frac{g'(N_s)}{u'(C)} \left( \frac{\partial C}{\partial C_s} \right) = \left[ \frac{g'(\cdot)}{u'(\cdot)} \right] \left( \frac{SC_s}{C} \right) = Z_s = MPL
\]

So, the efficiency condition can be expressed as

\[
g'(\cdot)/u'(\cdot) = Z_s(C/SC_s)
\]

To see how the monopoly distortions in the labor and goods markets are sources of inefficiency in our model, note that the wage setting equation (9) and constant expenditure shares (14) imply

\[
g'(\cdot)/u'(\cdot) = \left( \frac{1}{\mu_w}(W_s/P) \right) = \left( \frac{1}{\mu_w}(W_s/P_s)(P_s/P) \right) = \left( \frac{1}{\mu_w}(W_s/P_s)(C/SC_s) \right)
\]

Then use the price setting equation (11) to get

\[
g'(\cdot)/u'(\cdot) = \left( \frac{1}{\mu_w \mu_p} \right)Z_s(C/SC_s) = \left( \frac{1}{\mu} \right)Z_s(C/SC_s) < Z_s(C/SC_s)
\]
Proposition 7 (Benigno): Suppose a currency union consists of the two countries, A and B, described above. Then,

A. there is no full information policy rule that can achieve the optimal flexible wage/price solution union wide;
B. the optimal policy rule sets $\text{VAR}_{t-1}[(\omega_t - (1/3)z_{a,t}) - (2/3)z_{b,t} + a_t/\psi] = 0$
C. the optimal policy can also be characterized as $\text{VAR}_{t-1}[(1/3)p_{ap,t}^* + (2/3)p_{bp,t}^*] = \text{VAR}_{t-1}[(1/3)p_{af,t} + (2/3)p_{bf,t}] = 0$.

Proof:
A. This result, like Proposition 6, follows directly from Lemma 3.
B. From Proposition 1, Part A, the optimal monetary policy minimizes $J = \text{VAR}_{t-1}[\omega_t - z_{a,t} + a_t/\psi] + 2\text{VAR}_{t-1}[\omega_t - z_{b,t} + a_t/\psi]$
Expanding the variance terms, we have:

$J = 3\text{VAR}_{t-1}[\omega_t + a_t/\psi] + \text{VAR}_{t-1}[z_{a,t}] + 2\text{VAR}_{t-1}[z_{b,t}]$
- $2\text{COV}_{t-1}[\omega_t + a_t/\psi, z_{a,t}] - 4\text{COV}_{t-1}[\omega_t + a_t/\psi, z_{b,t}]$

$= 3\text{VAR}_{t-1}[\omega_t + a_t/\psi] + \text{VAR}_{t-1}[z_{a,t}] + 2\text{VAR}_{t-1}[z_{b,t}]$
- $6\text{COV}_{t-1}[\omega_t + a_t/\psi, z_{a,t}/3] - 6\text{COV}_{t-1}[\omega_t + a_t/\psi, 2z_{b,t}/3]$

$= 3\text{VAR}_{t-1}[\omega_t + a_t/\psi] + \text{VAR}_{t-1}[z_{a,t}] + 2\text{VAR}_{t-1}[z_{b,t}]$
- $6\text{COV}_{t-1}[\omega_t + a_t/\psi, z_{a,t}/3 + 2z_{b,t}/3]$

$= 3\text{VAR}_{t-1}[\omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3] + \text{VAR}_{t-1}[z_{a,t}] + 2\text{VAR}_{t-1}[z_{b,t}]$
- $3\text{VAR}_{t-1}[z_{a,t}/3 + 2z_{b,t}/3]$

$= 3\text{VAR}_{t-1}[\omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3] + (2/3)\text{VAR}_{t-1}[z_{a,t}] + (2/3)\text{VAR}_{t-1}[z_{b,t}]$
- $(4/3)\text{COV}_{t-1}[z_{a,t}/3, 2z_{b,t}/3]$

$= 3\text{VAR}_{t-1}[\omega_t + a_t/\psi - z_{a,t}/3 - 2z_{b,t}/3] + (2/3)\text{VAR}_{t-1}[z_{a,t} - z_{b,t}]$

Since the second variance term in the last expression does not depend on $\omega_t$, the optimal policy minimizes $J$ by setting the first variance term equal to zero.
C. We have
\[ \omega_t - (1/3)z_{a,t} - (2/3)z_{b,t} + a_t/\psi = (1/3)[\omega_t - z_{a,t} + a_t/\psi] + (2/3)[\omega_t - z_{b,t} + a_t/\psi] \]
\[ = (1/3)[\omega_t - y_{ap,t}^* - \psi^{-1}\log(\mu_t)] + (2/3)[\omega_t - y_{bp,t}^* - \psi^{-1}\log(\mu_t)] \]
\[ = (1/3)p_{ap,t}^* + (2/3)p_{bp,t}^* - \psi^{-1}\log(\mu_t) \]
(\text{where we have used } \omega_t = p_{s,t}^* + y_{s,t}^*)

So, B implies C.

Note that C says that the optimal policy eliminates uncertainty about a union-wide price index in which the country weights (one-third and two-thirds) depend on the sizes of sticky price sectors.

**Proposition 9 (Erceg, Henderson and Levin):** Suppose the economy consists of a fixed wage sector (denoted by “w”), a sticky price sector (denoted by “p”), and a flexible wage/price sector (denoted by “f”). Suppose productivity shocks are perfectly correlated across sectors (\(z_{p,t} = z_{w,t} = z_{f,t} = z_t\)). Then,

A. There is no rule, \(\omega_t(a_t, z_t)\), that makes \(y_{p,t} = y_{p,t}^*\) and \(y_{w,t} = y_{w,t}^*\) for all values of \(z_t\).

B. A policy that targets the aggregate price level (makes \(p_t = p_t^T\)) implies \(y_{p,t} = y_{p,t}^*\).

C. A policy that targets the aggregate wage rate (makes \(w_t = w_t^T\)) implies \(y_{w,t} = y_{w,t}^*\).

D. The optimal policy rule sets \(\text{VAR}_{t-1}[\omega_t - (1/2)z_t + a_t/\psi] = 0\)

E. The optimal policy can also be characterized as
\[ \text{VAR}_{t-1}[(1/2)p_{p,t}^* + (1/2)w_{w,t}^*] = \text{VAR}_{t-1}[(1/2)p_{f,t} + (1/2)w_{f,t}] = 0 \]

**Proof:**

A. Lemma 3 implies that: (1) to get the fixed price sector right, \(\omega_t\) must respond to both \(a_t\) and \(z_t\), and (2) to get the fixed wage sector right, \(\omega_t\) must respond to \(a_t\) alone.

B. The proof is similar to the proof of Proposition 3.

C. The proof is similar to the proof of Proposition 7.
D. From Proposition 1, Part A, the optimal monetary policy minimizes

\[ J = \text{VAR}_t[\omega_t - z_t + a_t/\psi] + \text{VAR}_t[\omega_t + a_t/\psi]. \]

Expanding the first variance term, we have

\[
J = 2\text{VAR}_t[\omega_t + a_t/\psi] + \text{VAR}_t[z_t] - 2\text{COV}_t[\omega_t + a_t/\psi, z_t] \\
= 2\text{VAR}_t[\omega_t + a_t/\psi] + \text{VAR}_t[z_t] - 4\text{COV}_t[\omega_t + a_t/\psi, z_t/2] \\
= 2\text{VAR}_t[\omega_t + a_t/\psi - z_t/2] + (1/2)\text{VAR}_t[z_t]
\]

Since the second variance term does not depend on \( \omega_t \), optimal policy minimizes \( J \) by setting the first variance term equal to zero.

E. We have

\[
\omega_t - (1/2)z_t + a_t/\psi = (1/2)[\omega_t - z_t + a_t/\psi] + (1/2)[\omega_t - a_t/\psi] \\
= (1/2)[\omega_t - y_{p,t}^* - \psi^{-1}\log(\mu)] + (1/2)[\omega_t - y_{w,t}^* + z_t - \psi^{-1}\log(\mu)] \\
= (1/2)p_{p,t}^* + (1/2)[w_{w,t}^* + \log p_t] - \psi^{-1}\log(\mu) \\
\text{(where we have used } \omega_t = p_{s,t}^* + y_{s,t}^* \text{ and } p_{s,t}^* = \log(\mu_p) + w_{s,t}^* - z_{s,t})
\]

So, \( \text{VAR}_t[\omega_t - (1/2)z_t + a_t/\psi] = 0 \)

\[ \Rightarrow \text{VAR}_t[(1/2)p_{p,t}^* + (1/2)w_{w,t}^*] = 0 \]