The Durable Information Monopolist

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1 Introduction

Information is a unique good as merely using it often gives it away. This ‘use it or lose it’ property of information seems to be quite general. For example, consider an agent with private information that an asset’s value exceeds its cost: attempting to purchase the asset reveals (a portion of) this information. Another example that has the same property, is an agent with private information about the presence of gold on public land. To the extent to which extracting that gold is observable, the agent cannot (fully) capitalize on her private information without others emulating her. Finally, Consider Churchill’s Dilemma during World War II. He owned the Nazi “Enigma” machine, which meant he could break the Nazi’s secret code. Using this, he learned that a major city, Coventry, was about to be heavily bombed. He chose not to act on this information and instead allowed Coventry to be flattened, because taking precautionary measures would inform the Nazi’s that they had broken their code, and make this information worthless.

What are the important common elements to these examples? An agent has private information. The optimal action depends on the private information and so actions can reveal the private information to others. They are all dynamic: you can buy assets over time, you extract gold over time, you accept the bombing of Coventry today to use the Enigma machine in the future. Finally, actions in each example are not perfectly observable: you can disguise your purchases of assets, you can sneak onto the public land to mine gold, Churchill could take some measures to protect Coventry without the Nazis necessarily deciding the English had broken their code.

We are certainly not the first to notice that attempting to act on private information lowers the value of that information when actions are (at least partially) observable. Grossman (1976) showed that in a common values model in which agents have exogenous private information about the value of a good to be traded then in Rational Expectations Equilibrium no trade takes place, thus no one can benefit from their private information. Grossman and Stiglitz (1980) endogenized the acquisition of private information and showed that without noise there does not exist an equilibrium. Kyle (1985) investigates a dynamic model of asset trading with one informed agent (an information monopolist) and noise traders, whose purchases are indistinguishable. Kyle (1985) shows that the information monopolist cannot fully extract the information rents, but is able to extract some by trading over multiple periods: essentially disguising his demand by ‘hiding’ amongst the noise traders.

Common to these examples is that the Information Monopolist cannot extract very much rent from any info he owns without giving it away. We wish to investigate this general
phenomenon in a simple model that we hope encompasses the basic features of a wide range of applications. The basic structure of our model is this: one agent has private, payoff relevant information. This agent must take an action in each of an infinite number of periods. This action is observable (perhaps with noise) to other agents. To highlight the common elements of these examples, we abstract from the details as much as possible, treating the payoff function like a black box (although we do provide some motivation for the payoffs chosen, and feel that the basic structure is quite general).

There is a strain of the repeated games with incomplete information literature that investigates when informed players can exploit their private information and when they cannot. In particular, the undiscounted constant sum case is well developed (see Aumann, Maschler, and Stearns (1995) for a thorough treatment). To fix ideas, consider a two player constant sum infinitely repeated game with no discounting in which each player has two strategies. Chance moves first and selects one of two payoff matrices, with chances \((q, 1 - q)\). Players cannot observe their payoffs, but actions are fully observable. Player 1 knows which payoff matrix has been selected.

Consider a one period game with the same payoff as the stage game, but in which Player 1 is not informed of the state. Call the value to Player 1 in this one period game \(u(q)\). Let \(cav u\) be the smallest concave function greater than or equal to \(u\). Aumann, Maschler, and Stearns show that the value of the original infinitely repeated game (say \(u_\infty\)) exists and equals \(cav u\) (Theorem 5.1 in Aumann, Maschler, and Stearns (1995)). Note that if \(cav u(q) = u(q)\) then Player 1’s optimal strategy is to play the game as if he did not know the state, i.e. to fully conceal his information. Clearly if \(u\) is a concave function, Player 1 will fully conceal his information for all \(q\).

We model the economic applications cited above as infinitely repeated games with incomplete information in which the stage games have strictly concave \(u\), which the results outlined above suggest are exactly the games in which Player 1 would like to conceal his information. We ask the following basic questions: How does an agent with private information optimally extract rents dynamically? To what extent are rents extracted, and how does this depend on patience (shades of Coase?) and noise?
2 The Model

For now we consider the simplest case: two players, two strategies, and two possible states of nature: \{H, L\}, which nature chooses with chances \{q, 1 − q\}. Player 1 is informed of the state. Neither observes the realized payoffs. Player 2 observes the actions of Player 1 with noise. They discount the future at interest rate \(r\).

More specifically, consider the following simple payoffs\(^1\):

\[
\begin{array}{ccc}
T & \ell & r \\
B & 0 & 0
\end{array}
\]

\[
\begin{array}{ccc}
T & \ell & r \\
B & 1 & 0
\end{array}
\]

Figure 1: In state \(H\) the payoffs are given by the left hand matrix, while in state \(L\) the payoffs are given by the right hand matrix.

In this example, \(u(q) = q(1 − q)\), and thus \(cav\) \(u = u\) and Player 1 cannot benefit from his information at all in the undiscounted game. That is, Player 1 will fully conceal his information.

In addition to adding noise and discounting, we work in continuous time. Each player continuously chooses a mixture chance. We seek to characterize Markovian Equilibria (ME), so let \(\alpha_q : [0, 1] \to [0, 1]\) be the state contingent probability that Player 1 plays \(T\) given \(q\), while \(\beta : [0, 1] \to [0, 1]\) is the probability that Player 2 plays \(\ell\) given \(q\).

Let \(Y\) be the total amount of \(T\) signals observed by Player 2. We model noisy observations by assuming \(Y\) evolves according to the following Brownian motions conditional on the strategy chosen by Player 1:

\[
\begin{align*}
T \text{ chosen: } dY &= dt + \sigma dW \\
B \text{ chosen: } dY &= \sigma dW
\end{align*}
\]

Thus, the variance (noise) is state independent. Given mixing \(\alpha_q\) by Player 1, we have

\[
dY = \alpha_q(q) dt + \sigma dW.
\]

Let Player 2 believe that Player 1 is mixing with state contingent chances \(\hat{\alpha}_q\). So that from the perspective of Player 2, Player 1 is mixing with chance:

\[
\alpha(q) = q\hat{\alpha}_H(q) + (1 − q)\hat{\alpha}_L(q)
\]

\(^1\)A motivation for these payoffs is the following: Player 1 and Player 2 are competing with Player 1 at a strategic disadvantage. Player 1 can only ‘win’ if he chooses a different strategy than Player 2 and that strategy turns out to be better given the current state of the world.
3 Equilibrium Characterization

Evolution of Beliefs. Player 1 must trade off flow profits and the rate at which information is revealed, when choosing $\alpha_H$ and $\alpha_L$. In equilibrium, player 2’s beliefs will be correct, but player 1 must also consider off the equilibrium path deviations when determining a best response. Thus, we need to know how player 2’s beliefs will evolve given any actual mixing probability chosen by player 1.

Consider the change in beliefs over some small interval of time $\Delta t$, then Bayes’ rule yields:

$$\Delta q = q(t + \Delta t) - q = \frac{q(1 - q)(f_H(\Delta Y) - f_L(\Delta Y))}{f_H(\Delta Y) + (1 - q)f_L(\Delta Y)}$$

where

$$f_\theta(\Delta Y) = \frac{1}{\sqrt{2\pi \Delta t}} e^{-\frac{1}{2\Delta t} (\Delta Y - \hat{\alpha}_\theta(q) \Delta t)^2}$$

and $\Delta Y$ equals the change in total signal $Y$ over the interval $\Delta t$. Algebraic Manipulation yields:

$$\Delta q = q(1 - q)(g_H(\Delta Y) - g_L(\Delta Y))$$

where $g_\theta(\Delta Y) = e^{-\hat{\alpha}_\theta(q)\Delta Y - \hat{\alpha}_\theta(q)^2 \Delta t/2}$. Taking a second order approximation to $g_\theta(\Delta Y)$, we have:

$$g_\theta(\Delta Y) \approx 1 + (\hat{\alpha}_\theta(q) \Delta Y - \frac{1}{2} \hat{\alpha}_\theta(q)^2 \Delta t) + \frac{1}{2} (\hat{\alpha}_\theta(q) \Delta Y - \frac{1}{2} \hat{\alpha}_\theta(q)^2 \Delta t)^2$$

Thus,

$$\Delta q \approx \frac{q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q)) \Delta Y}{1 + \alpha(q) \Delta Y}$$

$$\approx q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q)) \Delta Y [1 - \alpha(q) \Delta Y]$$

$$= q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))[\Delta Y - \alpha(q) \Delta t]$$

$$= q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))[\alpha_\theta(q) \Delta t - \alpha(q) \Delta t + \Delta W]$$

where we recall $\alpha(q) = q\hat{\alpha}_H(q) + (1 - q)\hat{\alpha}_L(q)$ and the last equality follows from $\Delta Y = \alpha_\theta(q) \Delta t + \Delta W$ and $\alpha$ is the actual mixing probability chosen by player 1. Then we have that:

$$E[\Delta q] \approx q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))(\alpha - \alpha(q)) \Delta t$$
and
\[ \text{Var}[\Delta q] = q^2(1 - q)^2(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))^2 \Delta t \]

Thus, \( q \) is an Ito process with drift, with
\[ dq = q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))(\alpha(q) - \alpha(q))dt + q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))d\hat{W} \]

Notice that the variance of player 2’s belief \( q \) is not a function of the actual \( \alpha(q) \) chosen by player 1, while the drift is increasing in the actual \( \alpha(q) \) chosen. This may seem at odds with beliefs being a martingale. However, notice that in equilibrium \( q \) will be a martingale from the perspective of player 2, since from player 2’s perspective \( E(\alpha(q)) = \alpha(q) \), which implies the expected drift is zero. However, since player 1 knows the actual state, beliefs will not be a martingale from the point of view of player 1. This should be expected: player 1 expects the true state of the world to be revealed slowly over time, and thus expects \( q \) to drift up in state \( H \) and down in state \( L \).

**Markovian Best Responses.** Note that Player 2’s actions cannot impact the evolution of the beliefs, as Player 2 has no information to reveal. Thus, in any ME Player 2 will simply myopically best respond to \( \alpha(q) \). We then have,

\[
\beta^*(q) = \begin{cases} 
0 & \forall \alpha(q) < 1 - q \\
n & \alpha(q) = 1 - q \\
1 & \forall \alpha(q) > 1 - q 
\end{cases}
\]

Given Player 2s beliefs \( \hat{\alpha}_\theta \) and her strategy \( \beta \), Player 1’s Bellman equation (via Ito’s Lemma) is\(^2\):

\[ rV_\theta(q) = \max_{\alpha(\theta) \in [0,1]} \Gamma_\theta(q, \alpha(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta) \]

where \( \Gamma_\theta(q, \alpha(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta) \equiv \]

\[
\pi_{\theta}(\alpha(q), \beta(q)) + q(1-q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))(\alpha(q) - \alpha(q))V_{\theta}'(q) + \frac{1}{2}q^2(1-q)^2(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))^2V_{\theta}''(q),
\]

\[ \pi_H(\alpha_H(q), \beta(q)) = \alpha_H(q)(1 - \beta(q)) \quad \text{and} \quad \pi_L(\alpha_L(q), \beta(q)) = (1 - \alpha_L(q))\beta(q).\]

Since \( \Gamma_\theta(q, \alpha(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta) \) is linear in \( \alpha(q) \) we have the following complementary slackness

\(^2\)We have suppressed the dependence of \( V_\theta \) on the \( \alpha \theta_{\alpha} \) and \( \beta \)
conditions:
\[
\frac{\partial \Gamma_{\theta}(q, \alpha_{\theta}(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta)}{\partial \alpha_{\theta}(q)} > 0 \quad \Rightarrow \quad \alpha_{\theta}(q) = 1
\]
\[
\frac{\partial \Gamma_{\theta}(q, \alpha_{\theta}(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta)}{\partial \alpha_{\theta}(q)} < 0 \quad \Rightarrow \quad \alpha_{\theta}(q) = 0
\]
\[
\alpha_{\theta}(q) \in (0, 1) \quad \Rightarrow \quad \frac{\partial \Gamma_{\theta}(q, \alpha_{\theta}(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta)}{\partial \alpha_{\theta}(q)} = 0
\]

where
\[
\frac{\partial \Gamma_H(q, \alpha_H(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta)}{\partial \alpha_H(q)} = 1 - \beta(q) + q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))V_H'
\]
\[
\frac{\partial \Gamma_L(q, \alpha_L(q), \hat{\alpha}_H, \hat{\alpha}_L, \beta)}{\partial \alpha_L(q)} = -\beta(q) + q(1 - q)(\hat{\alpha}_H(q) - \hat{\alpha}_L(q))V_L'
\]

A Markovian equilibrium (or Solution) is a tuple \((V_H, V_L, \beta, \alpha_H, \alpha_L)\) of five functions such that, given \(\hat{\alpha}_\theta = \alpha_\theta\) the four functions \((V_H, V_L, \alpha_H, \alpha_L)\) solves the Bellman Equation, and \(\beta\) is a best response to \(\alpha(q)\). That is, \(\beta\) is consistent with \(\beta^*\) given \(\hat{\alpha}_\theta = \alpha_\theta\).

The following result should be a straightforward application of Harris (1993).

**Proposition 1** A Markovian equilibrium exists with \(V_\theta \in C^2\).

**Lemma 1 (Boundary Values)** The following obtains at the boundaries: \(\beta(0) = 0, \beta(1) = 1, \alpha_H(0) = 1\) and \(\alpha_L(1) = 0\). Further the value functions are pinned: \(rV_H(0) = rV_L(1) = 1\) and \(rV_H(1) = rV_L(0) = 0\).

**Equilibrium Behavior.** In this section we characterize behavior in equilibrium. Keep in mind that in equilibrium \(\hat{\alpha}_\theta = \alpha_\theta\).

Intuitively, we should have \(\alpha_L \leq \alpha_H\). Notice that flow profits are increasing in \(\alpha_H(q)\) in State \(H\) and decreasing in \(\alpha_L(q)\) in State \(L\). However, we cannot \(a\ priori\) rule out \(V_L'(q) < 0 < V_H'(q)\) for some range of \(q\), so that there might be dynamic gains to setting \(\alpha_H(q) < \alpha_L(q)\), and so cannot immediately conclude that \(\alpha_L(q) < \alpha_H(q)\).

**Lemma 2 (Separation)** Player 1 is more likely to take an action when it is statically optimal to do so, that is \(\alpha_L(q) < \alpha_H(q)\) for all \(q \in (0, 1)\).

**Proof:** First we show that \(\alpha_L(q) \leq \alpha_H(q)\). Imagine Player 1 restricting himself to choosing the same mixing probability across both states. He cannot do better with this restriction than when optimizing conditional on the state. Now we have an equilibrium with \(\alpha_H(q) < \alpha_L(q)\)
Value functions are bounded away from 0 and 1: If Player 1 strictly mixes in state \( \bar{q} \), discounted payoffs of Player 1 must be higher under \( \bar{q} \). Player 1’s flow profits are strictly increased in each state. Thus, the unconditional expected change, player 1 is better off maximizing static profits, which implies \( \alpha (q) = 0 \) and player 2’s best response is \( \alpha (q) = 1 \) if \( \beta (q) \neq 0 \). But then \( \alpha (q) = 1 \) and player 2’s best response is \( \beta (q) = 1 \), a contradiction.

To see that the inequality must be strict, assume \( \alpha_H (q) = \alpha_L (q) \), then \( dq = 0 \), which implies that \( r V_L (q) = \max_{\theta (q)} \pi (\theta (q), \beta (q)) \). Thus, if \( \beta (q) \in (0, 1) \) then Player 1’s best response is: \( \alpha_H (q) = 1 \) and \( \alpha_L (q) = 0 \).

If instead \( \beta (q) = 1 \), then player 1’s best response is \( \alpha_L (q) = 0 \), since \( \alpha_H (q) \) was assumed equal to \( \alpha_L (q) \), then we have \( \alpha_H (q) = \alpha_L (q) = 0 \), but then \( \alpha (q) = 0 \) and player 2’s best response is \( \beta (q) = 0 \), a contradiction. If \( \beta (q) = 0 \), then player 1’s best response is \( \alpha_H (q) = 1 \), and thus for the conditional mixing probabilities to be equal, \( \alpha_L (q) = 1 \), but then \( \alpha (q) = 1 \) and player 2’s best response is \( \beta (q) = 1 \), a contradiction.

The intuition for this result is straightforward: if player 2 believes that player 1 is pooling, then he does not update his beliefs on the state of the world. But if player 2’s beliefs do not change, player 1 is better off maximizing static profits, which implies \( \alpha_H (q) \neq \alpha_L (q) \). Notice that this result implies that in the long run the true state is discovered with probability 1.

**Lemma 3** Value functions are bounded away from 0 and 1: \( 0 < r V_L (q) < 1 \) for all \( q \in (0, 1) \).

*Proof:* At the extremes, \( q = 0 \) and 1. Further value functions are continuous. Thus, there exist open intervals such that value functions are bounded away from zero near one extreme and bounded away from 1 near the other extreme on these intervals. The drift and variance of \( q \) are non zero and bounded. Thus, if we fix any \( T > 0 \), the expected time that \( q_t \) spends in each interval near the extremes is bounded away from 0. The result then follows from \( 0 < r < \infty \).

We now know that \( \alpha_L (q) < \alpha_H (q) \) for interior \( q \). What restrictions can we place on \( \beta \)?

**Lemma 4** If Player 1 strictly mixes in state \( H \), then \( \beta (q) < 1 \), while if Player 1 strictly mixes in state \( L \), then \( \beta (q) > 0 \).

*Proof Sketch:* We shall establish the result for \( \alpha_H (q) < 1 \), the \( \alpha_L (q) > 0 \) case is symmetric.

If \( \beta (q) = 1 \) and \( V_H' (q) \neq 0 \) then player 1’s best response in the high state is \( \alpha_H (q) \in \{ 0, 1 \} \), thus we can only have \( \alpha_H (q) < 1 \) and \( \beta (q) = 1 \) if \( V_H' (q) = 0 \). But \( V_H' (q) = 0 \) on any interval,
implies \( V_H''(q) = 0 \), which all together implies \( V_H(q) = 0 \), which contradicts value functions strictly positive for all interior \( q \).

Intuitively, Player 1 is better off the more Player 2 is convinced in the incorrect state of the world.

**Lemma 5 (Strict Monotonicity)** Value functions are strictly monotonic: \( V_H' < 0 \) and \( V_L' > 0 \).

*Proof*: Not done

Once this result is established we can show that Player 2 must strictly mix in equilibrium.

**Lemma 6** Player 2 strictly mixes for all \( q \in (0, 1) \) (i.e. \( \beta(q) \in (0, 1) \forall q \)).

*Proof*: If \( \beta(q) = 0 \) then \( \alpha_L(q) = 0 \). The complementary slackness conditions then imply:

\[
q(1 - q)(\alpha_H(q) - \alpha_L(q))V_H'(q) \leq 0
\]

but this violates \( \alpha_H(q) > \alpha_L(q) \) and \( V_H'(q) > 0 \).

If instead \( \beta = 1(q) \) then \( \alpha_H(q) = 1 \). The complementary slackness conditions then imply:

\[
q(1 - q)(\alpha_H(q) - \alpha_L(q))V_H'(q) \geq 0
\]

but this violates \( \alpha_H(q) > \alpha_L(q) \) and \( V_H'(q) < 0 \).

All together, the above lemmas rule out all but the following two cases:

**Strict Mixing** Player 1 strictly mixes in each state of the world and player 2 strictly mixes:

\( \beta(q) \in (0, 1) \) and \( 0 < \alpha_L(q) < \alpha_H(q) < 1 \).

**Partial Mixing** Player 1 strictly mixes in one state of the world and chooses a pure strategy in the other, while player 2 strictly mixes: \( \beta(q) \in (0, 1) \) and either \( 0 = \alpha_L(q) < \alpha_H(q) < 1 \) or \( 0 < \alpha_L(q) < \alpha_H(q) = 1 \).

**Strict Mixing.** First, we consider the strict mixing case. In this is the case \( \partial \Gamma_\theta / \partial \alpha_\theta = 0 \) (by the complementary slackness conditions), so that,

\[
(1 - \beta(q)) + q(1 - q)(\alpha_H(q) - \alpha_L(q))V_H'(q) = 0 \quad \text{State H}
\]

\[
-\beta(q) + q(1 - q)(\alpha_H(q) - \alpha_L(q))V_L'(q) = 0 \quad \text{State L}
\]
Eliminating $\beta$ from the above yields:

$$\alpha_H(q) - \alpha_L(q) = [q(1-q)(V'_L(q) - V'_H(q))]^{-1} \quad (2)$$

The drift and variance are both (increasing) functions of $\alpha_H(q) - \alpha_L(q)$. Thus, characterizing this difference is critical to understanding the evolution of $q$ in equilibrium.

**Lemma 7** Assuming strict mixing, $q = 1/2$ is a critical point of $\alpha_H(q) - \alpha_L(q)$, a local minimum if $V''_H \geq 0$ and $V''_L \leq 0$.

Equation (2) implies that $\alpha_H(q) - \alpha_L(q)$ is maximized (minimized) when $h(q) \equiv q(1-q)(V'_L(q) - V'_H(q))$ is minimized (maximized). We compute:

$$h'(q) = (1-2q)(V'_L(q) - V'_H(q)) + q(1-q)(V''_L(q) - V''_H(q))$$

By symmetry, $V_H(q) = V_L(1-q)$ and thus, $V''_H(1/2) = V''_L(1/2)$, which implies $h'(1/2) = 0$.

$$h''(q) = 2(1-2q)(V''_L(q) - V''_H(q)) - 2(V'_L(q) - V'_H(q)) + q(1-q)(V'''_L(q) - V'''_H(q))$$

so that:

$$h''(1/2) = -2(V'_L(1/2) - V'_H(1/2)) + (V'''_L(1/2) - V'''_H(1/2))/4$$

Since $V'_L > 0$ and $V'_H < 0$, $h''(1/2) < 0$ follows from the assumption on the third derivatives of $V_\theta$. Thus, $q = 1/2$ is a local maximum of $h(q)$ and so a local minimum of $\alpha_H(q) - \alpha_L(q)$.

□

**Lemma 8** Whenever strict mixing obtains, $V'_L(q) - V'_H(q) \geq \max\{1/q^2, 1/(1-q)^2\}$.

**Proof:** $\beta(q) \in (0,1) \Rightarrow \alpha(q) = 1 - q$, which after some algebraic manipulation yields:

$$\alpha_H(q) - \alpha_L(q) = \frac{1-q - \alpha_L(q)}{q}$$

combining this with equation (2) yields:

$$\alpha_L(q) = 1 - q - \frac{1}{(1-q)(V'_L(q) - V'_H(q))} \geq 0$$

since $\alpha_L(q)$ is a probability. Further algebraic manipulation yields: $V'_L(q) - V'_H(q) \geq (1-q)^{-2}$.
For the other lower bound, manipulate $\alpha(q) = 1 - q$ to get:

$$\alpha_H(q) = \frac{1 - q}{q}(1 - \alpha_L(q))$$

Combining this with the above equation for $\alpha_L$ yields:

$$\alpha_H(q) = 1 - q + [q(V'_L(q) - V'_H(q))]^{-1} \leq 1$$

manipulation then yields: $V'_L(q) - V'_H(q) \geq q^{-2}$. □

Using equation (2) plus the implied $\beta$ from the $\Gamma'_{\theta} = 0$ equations combined with the Bellman Equation yields the following system of second order differential equations for the value functions:

$$rV_H(q) = -(1 - q) \frac{V'_H(q)}{V'_L(q) - V'_H(q)} + \frac{V''_H(q)}{2(V'_L(q) - V'_H(q))^2}$$

$$rV_L(q) = q \frac{V'_L(q)}{V'_L(q) - V'_H(q)} + \frac{V''_L(q)}{2(V'_L(q) - V'_H(q))^2}$$

We can combine these two equations in order to provide a link between this discounted noisy game and the undiscounted game. Let $V(q) \equiv qV_H + (1 - q)V_L$, so that the above two equations imply:

$$rV(q) = q(1 - q) + \frac{qV''_H(q) + (1 - q)V''_L(q)}{2(V'_L(q) - V'_H(q))^2}$$

(3)

Note that the first term is the maximum perfectly concealing payoff. That is the average payoff in the undiscounted game without noise, in which player 1 reveals no information. The second term can then be interpreted as Player 1’s information rent. This in turn implies that this second term is positive.

Toward reducing our system to one second order differential equation define $\Delta(q) \equiv V_L(q) - V_H(q)$. Substituting the definitions of $V$ and $\Delta$ into the $\Gamma'_{\theta} = 0$ equations yields:

$$r\Delta(q) = \frac{qV'_L(q) + (1 - q)V'_H(q)}{\Delta'(q)} + \frac{\Delta''(q)}{2\Delta'(q)^2}$$

$$= q + \frac{V'_H(q)}{\Delta'(q)} + \frac{\Delta''(q)}{2\Delta'(q)^2}$$

To eliminate $V'_H$ from the above equation, note that $V_H = V - (1 - q)\Delta$, thus $V'_H = V' -
\[(1 - q)\Delta' + \Delta, \text{ so that:}\]

\[r\Delta(q) = 2q - 1 + \frac{V'(q) + \Delta}{\Delta'(q)} + \frac{\Delta''(q)}{2\Delta'(q)^2}\]  \hspace{1cm} (4)

**Lemma 9** Near the extremes \(q = 0 \text{ or } 1\), we cannot have a strict mixing equilibrium.

**Proof:** Solve equation (4) for \(V'\) to get:

\[V'(q) = r\Delta(q)\Delta'(q) + (1 - 2q)\Delta'(q) - \Delta - \frac{\Delta''(q)}{2\Delta'(q)}\]

then integrate the above to recover \(V(q)\) noting that:

\[
\int_0^q r\Delta(q)\Delta'(t)dt = \left(r\Delta(q)^2 - r\right)/2
\]

\[
\int_0^q [(1 - 2t)\Delta'(t) - \Delta(t)]dt = (1 - 2q)\Delta(q) + 1 + \int_0^q \Delta(t)dt
\]

\[
- \int_0^q \frac{\Delta''(t)}{2\Delta'(t)}dt = \frac{1}{2} \log \frac{\Delta'(0)}{\Delta'(q)}
\]

Putting these together yields:

\[V(q) = 1 + \frac{r}{2}(\Delta(q)^2 - 1) + (1 - 2q)\Delta(q) + \int_0^q \Delta(t)dt + \frac{1}{2} \log \frac{\Delta'(0)}{\Delta'(q)}\]

Now note that all of these terms are finite, save \(\log \Delta'(0)\), which is unbounded by Lemma 8, which implies \(V(q)\) unbounded, a contradiction. \(\square\)

We can further manipulate the FoCs to obtain a single (albeit nasty) uncoupled differential equation for \(\Delta\). First rearrange (3) to get:

\[rV(q) = q(1 - q) + \frac{V''(q) + 2\Delta'(q)}{2(\Delta'(q))^2}\]

Then differentiate to get:

\[rV'(q) = 1 - 2q + \frac{\Delta'(q)(V'''(q) + 2\Delta''(q)) + 2\Delta''(q)(V''(q) - q\Delta'(q))}{2(\Delta'(q))^3}\]  \hspace{1cm} (5)
Now solve equation (4) for $V'$ and then twice differentiate to get:

\[
V'(q) = \Delta'(q)(r\Delta(q) - 2q + 1) - \Delta(q) - \frac{\Delta''(q)}{2\Delta'(q)}
\]

\[
V''(q) = \Delta''(q)(r\Delta - 2q + 1) + \Delta'(q)(r\Delta'(q) - 2) - \Delta'(q) - \frac{1}{2}\left(\frac{\Delta''(q)}{\Delta'(q)}\right)'
\]

\[
V'''(q) = \Delta'''(q)(r\Delta(q) - 2q + 1) + 2\Delta''(q)(r\Delta'(q) - 2) + r\Delta'(q)\Delta''(q) - \Delta''(q) - \frac{1}{2}\left(\frac{\Delta''(q)}{\Delta'(q)}\right)''
\]

Substituting these into (5) yields a fourth order differential equation in $\Delta$ alone.

**Partial Mixing.** Note that we always require Player 2 indifference, which means $\alpha(q) = 1 - q$. There are two sub partial mixing cases. One valid when $q < 1/2$ and one valid when $q > 1/2$.

**Lemma 10** If $q > 1/2$ then we cannot have $\alpha_H(q) = 1$, while if $q < 1/2$ we cannot have $\alpha_L(q) = 0$.

**Proof:** If $\alpha_H(q) = 1$ then $\alpha(q) > q$, thus we cannot have $\alpha(q) = 1 - q$ if $q > 1/2$, as required for player 2 indifference. Likewise if $\alpha_L(q) = 0$ and $q < 1/2$ violated $\alpha(q) = 1/2$. □

Now, consider the case $\alpha_H(q) = 1$ and $\alpha_L(q) > 0$ and $q < 1/2$ (the other Partial Mixing case is symmetric). Combining $\alpha(q) = 1 - q$ and $\alpha_H(q) = 1$ yields: $\alpha_L(q) = (1 - 2q)/(1 - q)$ and thus $\alpha_H(q) - \alpha_L(q) = q/(1 - q)$. Substituting these expressions into the FoC for $\alpha_L(q)$ yields:

\[
\beta(q) = q^2 V_L''(q)
\]

Substituting the above into the Bellman equation and simplifying yields the following coupled second order differential equations:

\[
rV_H(q) = 1 - q^2 V_L'(q) + q^3 V_H'(q) + q^4 V_H''(q)/2
\]

\[
rV_L(q) = q^3 V_H'(q) + q^4 V_L'(q)/2
\]

The general solution to the $V_L$ equation is:

\[
V_L(q) = c_1 L e^{\sqrt{2r}/q} + c_2 L e^{-\sqrt{2r}/q}
\]

Since we know strict mixing cannot obtain at the extremes, and we know an equilibrium exists, this case must obtain near $q = 0$. Near $q = 0$ (and symmetrically the other partial
mixing case obtains near \( q = 1 \), boundedness implies \( c_L^1 = 0 \). Given this solution we have:

\[
V_H(q) = \frac{1}{r} + c_H^1 e^{\sqrt{2r}/q} + c_H^2 e^{-\sqrt{2r}/q} - \frac{c_L^2}{q\sqrt{2r}} e^{-\sqrt{2r}/q}
\]

Again boundedness implies \( c_H^1 = 0 \). So all together we have the following value functions and their derivatives near \( q = 0 \):

\[
\begin{align*}
V_L(q) &= c_L e^{-\sqrt{2r}/q} \\
V_H(q) &= \frac{1}{r} + c_L e^{-\sqrt{2r}/q} - \frac{c_L}{q\sqrt{2r}} e^{-\sqrt{2r}/q} \\
V_L'(q) &= \frac{\sqrt{2r}}{q^2} c_L e^{-\sqrt{2r}/q} \\
V_H'(q) &= \frac{\sqrt{2r}}{q^2} c_H e^{-\sqrt{2r}/q} - \frac{\sqrt{2r}}{q^2} c_L e^{-\sqrt{2r}/q} \\
V_L''(q) &= \frac{1}{q^3} e^{-\sqrt{2r}/q} \left[ \frac{2r}{q} - 2\sqrt{2r} \right] \left[ \sqrt{2r} c_H - \frac{c_L}{\sqrt{2r}} \right] + \frac{c_L}{q^4} e^{-\sqrt{2r}/q}
\end{align*}
\]

Since, \( \lim_{q \to 0} k_1 q^{-N} e^{-k_2/q} = 0 \) for all constants \( k_1, k_2, N > 0 \) we have that \( \lim_{q \to 0} V_H'(q) = \lim_{q \to 0} V_L''(q) = 0 \).

By inspecting the dominant terms, \( V_L'(q) > 0 \) and \( V_H'(q) < 0 \) for \( q \) sufficiently close to 0. Then \( V_H''(q) = V_L''(1 - q) \) would yield \( V_L''(q) < 0 \) and \( V_H''(q) > 0 \) for \( q \) sufficiently close to 1. Note that the sign of \( V_H'' \) near each extreme suggests that we may have \( V_L'' \leq 0 \) and \( V_H'' \geq 0 \) which would in turn imply that \( \alpha_H(q) - \alpha_L(q) \) is minimized at \( q = 1/2 \) by Lemma 7.

These functional forms imply that \( \lim_{q \to 0} \beta(q) = 0 \) and \( \lim_{q \to 1} \beta(q) = 1 \) and that \( \beta \) is increasing: convex near \( q = 0 \) and concave near \( q = 1 \).

**Lemma 11** We cannot have partial mixing at \( q = 1/2 \).

**Proof:** Let \( V_L \) be the value function that obtains above 1/2 and \( V_H \) be the value function that obtains below 1/2. Symmetry implies that \( V_L(1/2) = V_H(1/2) \). At any switching boundary we require value matching \( (V_L(1/2) = V_H'(1/2)) \), smooth pasting \( (V_L'(1/2) = V_H''(1/2)) \), and super smooth pasting \( (V_L''(1/2) = V_H''(1/2)) \): three linear equations in two unknowns, \( c_L \) and \( c_H \). One can show using the above formulae for \( V_L \) and \( V_H \) that these cannot be satisfied simultaneously save for knife edged \( r \).

Note that since we cannot have strict mixing at the extremes and we cannot have partial mixing at \( q = 1/2 \), we suspect the equilibrium involves a cutoff \( \bar{q} < 1/2 \) such that strict
mixing obtains on \((\bar{q}, 1 - \bar{q})\) and partial mixing obtains at the extremes.

References


