Matching with Multiple Applications Revisited*

James Albrecht[†]
Georgetown University

Pieter Gautier
Erasmus University
Tinbergen Institute

Serene Tan
University of Pennsylvania

Susan Vroman
Georgetown University

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Abstract

This note corrects the matching function proposed in Albrecht, Gautier and Vroman (2003a). It also verifies that the limiting result given in that note is correct.

1 Introduction

Albrecht, Gautier, and Vroman (2003a) (hereafter AGV 2003a) proposed a generalization of the urn-ball matching function allowing for more than one application per worker. Suppose there are u unemployed workers and v vacancies. Each unemployed worker submits a applications with $a \in \{1, 2, ..., v\}$ given. These applications are randomly distributed across the v vacancies with the proviso that any particular worker sends at most one application to any particular vacancy. Each vacancy (among those that received at least one application) then chooses one application at random and offers that applicant a job. A worker may get more than one offer. In that case, the worker accepts one of the offers at random.

Let $M(u, v; a)$ be the expected number of matches, i.e., the expected number of accepted offers. AGV (2003a) presents expressions for $M(u, v; a)$ and for $m(\theta; a) \equiv \lim\limits_{u,v \to \infty, v/u=\theta} \frac{M(u,v;a)}{u}$. Tan (2003) points out that the AGV

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[†]Corresponding author: Department of Economics, Georgetown University, Washington DC 20057; tel: 202 687 6105; FAX: 202 687 6102; e-mail: albrecht@georgetown.edu

(2003a) expression for $M(u,v;a)$ is incorrect for $a \in \{2,...,v-1\}$, u and v finite, and presents a corrected expression. Albrecht, Gautier, and Vroman (2003b) (hereafter AGV 2003b) presents a corrected expression for $M(u,v;2)$ and proves that the AGV (2003a) expression for $m(\theta;a)$ is correct. This note summarizes the independently derived results of AGV (2003b) and Tan (2003).[1]

The problem in the finite case can be understood when $a=2$. Consider any vacancy to which an unemployed worker applies. The number of competitors the worker has at this vacancy is $bin(u-1,\frac{2}{v})$. One can then compute the probability that the worker fails to receive an offer at this vacancy. Similarly, the number of competitors at the other vacancy to which this worker applies is $bin(u-1,\frac{2}{v})$. Again, one can compute the probability that the worker fails to receive an offer from this vacancy. The probability that a worker receives at least one offer equals 1 minus the probability he or she receives no offers. The mistake in AGV (2003a) was to assume (implicitly) that the probability a worker receives no offers equals the probability that his first application doesn't generate an offer times the probability that his second application doesn't generate an offer. However, the indicator random variables, "first application leads to an offer" and "second application leads to an offer" are not independent. Equivalently, the numbers of competitors that a worker has at the 2 vacancies are not independent. Note that this problem does not arise when $a=1$ or $a=v$.

2 The Finite Case

Consider $a=2$. Then $M(u,v;2)=u\left(1-\Psi\right)$, where

$$\Psi=\sum_{i=0}^{u-1}\sum_{j=0}^{u-1}\Delta_1(i)\Delta_2(i,j)\tfrac{j}{j+1}\tfrac{i}{i+1}$$

is the probability that neither of a worker's applications is successful. The term

$$\Delta_1(i)=\tbinom{u-1}{i}\left(\tfrac{2}{v}\right)^i\left(1-\tfrac{2}{v}\right)^{u-1-i}$$

is the probability that the worker has i competitors at the first vacancy to which he applies, and

$$\Delta_2(i,j)=\sum_z\binom{i}{z}\binom{u-1-i}{j-z}\left(\frac{1}{v-1}\right)^z\left(1-\frac{1}{v-1}\right)^{i-z}\left(\frac{2}{v-1}\right)^{j-z}\left(1-\frac{2}{v-1}\right)^{u-1-i-(j-z)}$$

[1] See those two papers for details. Philip (2003) derives similar results.

is the conditional probability that the worker has j competitors at the second vacancy to which he applies given i. The summation over z in the expression for $\Delta_2(i,j)$ accounts for the fact that there may be some competitors who apply to both of the vacancies to which the worker in question applies. The presentation given here is essentially that of Tan (2003). AGV (2003b) derives the joint probability distribution for i and j directly. Of course, since $P[I=i, J=j] = \Delta_1(i)\Delta_2(i,j)$ the two approaches are equivalent. Details are given in our two papers.

Now consider any fixed $a \in \{2, ..., v-1\}$. Then $M(u,v;a) = u\left(1-\Psi\right)$, where

$$\Psi = \sum_i \sum_j \sum_k ... \sum_l \Delta_1(i)\Delta_2(i,j)\Delta_3(i,j,k)...\Delta_a(i,j,k,...,l)\tfrac{l}{l+1}\cdots\tfrac{k}{k+1}\tfrac{j}{j+1}\tfrac{i}{i+1}.$$

Here $\Delta_3(i,j,k)$ is the conditional probability that the worker has k competitors at the third vacancy to which he applies given i and j, ..., and $\Delta_a(i,j,k,...,l)$ is the conditional probability that the worker has l competitors at the last vacancy to which he applies given $i,j,k,...$ Expressions for the conditional probabilities are given in Tan (2003).

3 The Limiting Case

The above formula for $M(u,v;a)$, although complicated, reduces in the limit to the simple expression given in AGV (2003a), namely,

$$m(\theta;a) \equiv \lim_{u,v\to\infty, v/u=\theta} \tfrac{M(u,v;a)}{u} = 1 - \left(1 - \tfrac{\theta}{a}\left(1 - e^{-\frac{a}{\theta}}\right)\right)^a.$$

The derivation of the above expression is simplest to explain in the case of $a=2$. The key is to show that in the limit, I and J are independent, so that $P[I=i, J=j] = P[I=i]P[J=j]$. The algebra underlying this result uses the fact that for large u and v, the probability that any one worker will compete with another worker on more than one vacancy at a time is close to zero. Since the marginal distributions for I and J are each $bin(u-1, \frac{2}{v})$, we use the standard result on the Poisson as the limit of a binomial to show that

$$\lim_{u,v\to\infty, v/u=\theta} P[I=i]P[J=j] = h(i)h(j)$$

where

$$h(x) = \tfrac{2^x \theta^{-x} e^{-2/\theta}}{x!}.$$

Then, we have

$$\begin{aligned}\lim_{u,v\to\infty, v/u=\theta} \tfrac{M(u,v;a)}{u} &= 1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\frac{i}{i+1})(\frac{j}{j+1})h(i)h(j) \\ &= 1-\left(1-\tfrac{\theta}{2}\left(1-e^{-\frac{2}{\theta}}\right)\right)^2.\end{aligned}$$

To extend the limiting argument from the case of $a = 2$ to the general case of $a \in \{2, ..., A\}$, where A is an arbitrary (but fixed) number of applications, we need to show that in the limit the probability that any competitor applies to two or more of the vacancies to which an individual has applied is zero. The intuition is that in a large labor market, the outcome of a worker's application to any one vacancy tells us next to nothing about whether or not his other applications will succeed. The derivation is basically the same as the one used for $a = 2$. We show that the the numbers of competitors at the a vacancies to which the worker applies are approximately independently and identically distributed $bin(u-1, \frac{a}{v})$ random variables. Then, in the limit, we have that the joint probability is the product of a independent Poissons, each with parameter a/θ. Taking the limit as $u, v \to \infty$ with $v/u = \theta$ gives the AGV (2003a) expression for $m(\theta; a)$. The details are given in AGV (2003b).

References:

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